Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann
Series: Australian National University, Canberra
Adviser: M. F. Newman

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Dichotomies in Stability Theory

Springer-Verlag
Berlin Heidelberg New York 1978

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AMS Subject Classifications (1970): 34D05, 58F15

ISBN 3-540-08536-X Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-08536-X Springer-Verlag New York Heidelberg Berlin

Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2140/3140-543210

PREFACE

Several years ago I formed the view that dichotomies, rather than Lyapunov's characteristic exponents, are the key to questions of asymptotic behaviour for non-autonomous differential equations. I still hold that view, in spite of the fact that since then there have appeared many more papers and a book on characteristic exponents. On the other hand, there has recently been an important new development in the theory of dichotomies. Thus it seemed to me an appropriate time to give an accessible account of this attractive theory.

The present lecture notes are the basis for a course given at the University of Florence in May, 1977. I am grateful to Professor R. Conti for the invitation to visit there and for providing the incentive to put my thoughts in order. I am also grateful to Mrs Helen Daish and Mrs Linda Southwell for cheerfully and carefully typing the manuscript.

CONTENTS

1. STABILITY

A *dichotomy*, *exponential* or *ordinary*, is a type of conditional stability. Let us begin, then, by recalling some facts about unconditional stability.

The classical definitions of stability and asymptotic stability, due to Lyapunov, are well suited for the study of autonomous differential equations. For non-autonomous equations, however, the concepts of uniform stability and uniform asymptotic stability are more appropriate.

Let $\tilde{x}(t)$ be a solution of the vector differential equation

$$x' = f(t, x) \tag{1}$$

which is defined on the half-line $0 \le t < \infty$. The solution $\tilde{x}(t)$ is said to be *uniformly stable* if for each $\varepsilon > 0$ there is a corresponding $\delta = \delta(\varepsilon) > 0$ such that any solution $x(t)$ of (1) which satisfies the inequality $|x(s) - \tilde{x}(s)| < \delta$ for some $s \ge 0$ is defined and satisfies the inequality $|x(t) - \tilde{x}(t)| < \varepsilon$ for all $t \ge s$. It is said to be *uniformly asymptotically stable* if in addition there is a $\delta_0 > 0$ and for each $\varepsilon > 0$ a corresponding $T = T(\varepsilon) > 0$ such that if $|x(s) - \tilde{x}(s)| < \delta_0$ for some $s \ge 0$ then $|x(t) - \tilde{x}(t)| < \varepsilon$ for all $t \ge s + T$.

We will be interested in the application of these notions to the linear differential equation

$$x' = A(t)x \ , \tag{2}$$

where $A(t)$ is a continuous $n \times n$ matrix function for $0 \le t < \infty$. Let $X(t)$ be a fundamental matrix for (2). It may be shown without difficulty that the solution $x = 0$ of (2) is uniformly stable if and only if there exists a constant $K > 0$ such that

$$|X(t)X^{-1}(s)| \le K \quad \text{for} \quad 0 \le s \le t < \infty \ .$$

It is uniformly asymptotically stable if and only if there exist constants $K > 0$, $\alpha > 0$ such that

$$|X(t)X^{-1}(s)| \le Ke^{-\alpha(t-s)} \quad \text{for} \quad 0 \le s \le t < \infty \ .$$

For example, the scalar differential equation

$$x' = -(t + 1)^{-1}x \ ,$$

which has the fundamental solution $x(t) = (t + 1)^{-1}$, is uniformly stable and asymptotically stable, but not uniformly asymptotically stable.

The scalar differential equation

$$x' = \{\sin \log(t + 1) + \cos \log(t + 1) - \alpha\}x \ ,$$

which has the fundamental solution

$$x(t) = \exp\{(t + 1) \sin \log(t + 1) - \alpha t\} \ ,$$

is asymptotically stable and even 'exponentially stable' if $\alpha > 1$, but not uniformly stable if $\alpha < 2^{\frac{1}{2}}$.

Uniform asymptotic stability has the important property of being preserved under small perturbations of the coefficient matrix. This follows from

PROPOSITION 1. *Suppose* $A(t)$ *is a continuous matrix function on an interval* J *and the fundamental matrix* $X(t)$ *of the equation* (2) *satisfies the inequality*

$$|X(t)X^{-1}(s)| \le Ke^{\alpha(t-s)} \quad \textit{for} \quad t \ge s \ .$$

If $B(t)$ *is a continuous matrix function such that* $|B(t)| \le \delta$ *for all* $t \in J$ *then the fundamental matrix* $Y(t)$ *of the perturbed equation*

$$y' = [A(t) + B(t)]y \tag{3}$$

satisfies the inequality

$$|Y(t)Y^{-1}(s)| \le Ke^{\beta(t-s)} \quad \textit{for} \quad t \ge s \ ,$$

where $\beta = \alpha + \delta K$.

Proof. If $y(t)$ is any solution of (3) then, by the variation of constants formula,

$$y(t) = X(t)X^{-1}(s)y(s) + \int_s^t X(t)X^{-1}(u)B(u)y(u)du \ .$$

Hence, for $t \ge s$,

$$|y(t)| \le Ke^{\alpha(t-s)}|y(s)| + K\int_s^t e^{\alpha(t-u)}|B(u)||y(u)|du \ .$$

Thus the scalar function $w(t) = e^{-\alpha t}|y(t)|$ satisfies the inequality

$$w(t) \le Kw(s) + K\int_s^t |B(u)|w(u)du \quad \text{for} \quad t \ge s \ .$$

By Gronwall's inequality, this implies

$$w(t) \le Kw(s) \exp K\int_s^t |B(u)|du \quad \text{for} \quad t \ge s \ .$$

Consequently

$$|y(t)| \le K|y(s)|e^{\beta(t-s)} \quad \text{for} \quad t \ge s \ .$$

Taking $y(t) = Y(t)Y^{-1}(s)\xi$, where ξ is an arbitrary constant vector, we obtain the result.

Uniform stability, on the other hand, is preserved under absolutely integrable perturbations of the coefficient matrix. The following proposition may be established in a similar manner to the preceding one.

PROPOSITION 2. *Suppose* $A(t)$ *is a continuous matrix function on an interval* J *and the fundamental matrix* $X(t)$ *of the equation* (2) *satisfies the inequality*

$$|X(t)X^{-1}(s)| \le K \quad \textit{for} \quad t \ge s \ .$$

If $B(t)$ *is a continuous matrix function such that* $\int_J |B(t)|dt \le \delta$ *then the fundamental matrix* $Y(t)$ *of the perturbed equation* (3) *satisfies the inequality*

$$|Y(t)Y^{-1}(s)| \le L \quad \textit{for} \quad t \ge s \ ,$$

where $L = Ke^{\delta K}$.

For autonomous linear differential equations stability can be characterised in terms of the eigenvalues of the coefficient matrix. The equation

$$x' = A_0 x$$

is (uniformly) asymptotically stable if and only if all eigenvalues of the constant matrix A_0 have negative real part. It is (uniformly) stable if and only if all eigenvalues of A_0 have non-positive real part and those with zero real part are 'semisimple'.

We will now show by an example that this type of characterisation is no longer possible for non-autonomous equations. Take $A(t) = U^{-1}(t)A_0U(t)$, where

$$A_0 = \begin{pmatrix} -1 & -5 \\ 0 & -1 \end{pmatrix} , \quad U(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} .$$

Evidently $A(t)$ has both eigenvalues -1 for every t . But the corresponding equation (2) has the fundamental matrix

$$X(t) = \begin{pmatrix} e^t(\cos t + \tfrac{1}{2}\sin t) & e^{-3t}(\cos t - \tfrac{1}{2}\sin t) \\ e^t(\sin t - \tfrac{1}{2}\cos t) & e^{-3t}(\sin t + \tfrac{1}{2}\cos t) \end{pmatrix} ,$$

and is consequently unstable.

Thus eigenvalues fail as a general theoretical tool in the non-autonomous case. Nevertheless we will show that it is possible to salvage something.

PROPOSITION 3. *If* A *is an* $n \times n$ *matrix such that*

(i) every eigenvalue of A *has real part* $\leq \alpha$,

(ii) $|A| \leq M$,

and if $0 < \varepsilon < 2M$ *then*

$$|e^{tA}| \leq (2M/\varepsilon)^{n-1} e^{(\alpha+\varepsilon)t} \quad \textit{for} \quad t \geq 0 .$$

Proof. We show first that

$$|e^{tA}| \leq e^{\alpha t} \sum_{k=0}^{n-1} (2tM)^k/k! \text{ for } t \geq 0 . \tag{4}$$

Assume initially that A has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $p(\lambda)$ be the polynomial of degree $< n$ which takes the same values as $\phi(\lambda) = e^{t\lambda}$ at the points $\lambda_1, \ldots, \lambda_n$. Then, by the theory of functions of a matrix, $p(A) = e^{tA}$. If we represent $p(\lambda)$ by Newton's interpolation formula in the form

$$p(\lambda) = c_1 + c_2(\lambda - \lambda_1) + c_3(\lambda - \lambda_1)(\lambda - \lambda_2) + \ldots + c_n(\lambda - \lambda_1) \ldots (\lambda - \lambda_{n-1})$$

then the coefficients are given by $c_1 = \phi(\lambda_1)$ and, for $k > 1$,

$$c_k = \int_0^1 \int_0^{t_1} \ldots \int_0^{t_{k-2}} \phi^{(k-1)}[\lambda_1 + (\lambda_2 - \lambda_1)t_1 + \ldots + (\lambda_k - \lambda_{k-1})t_{k-1}]dt_1 \ldots dt_{k-1} .$$

Here the argument of $\phi^{(k-1)}$ belongs to the convex hull of $\lambda_1, \ldots, \lambda_n$ and hence to the half-plane $\mathcal{R}\lambda \leq \alpha$. Since $\phi^{(k-1)}(\lambda) = t^{k-1}e^{t\lambda}$ it follows that, if $t \geq 0$,

$$|c_k| \leq e^{\alpha t} t^{k-1}/(k-1)\ !$$

Also, for every eigenvalue λ_k we have $|\lambda_k| \leq |A|$ and hence $|A - \lambda_k I| \leq 2|A|$. Substituting these estimates in the formula for $p(A)$ we obtain (4). If the eigenvalues of A are not distinct then A is the limit of a sequence of matrices A_ν with distinct eigenvalues and by letting $\nu \to \infty$ we see that the inequality (4) holds also in the general case.

Now set

$$g_n(t) = e^{-\eta t} \sum_{k=0}^{n} t^k/k!\ ,$$

where $0 < \eta < 1$. Then $g_n(t) \to 0$ as $t \to \infty$, $g_n(0) = 1$, and

$$g_n' = g_{n-1} - \eta g_n \quad (n \geq 1) .$$

Thus $g_n'(0) = 1 - \eta > 0$ and $g_n(t)$ assumes its maximum value in the interval $[0, \infty]$ at an interior point. At this point $g_n = \eta^{-1} g_{n-1}$. Therefore $\mu_n = \max g_n(t)$ satisfies $\mu_n \leq \eta^{-1}\mu_{n-1}$. Since $\mu_0 = 1$ it follows that $\mu_n \leq \eta^{-n}$. Hence if $0 < \varepsilon < 2M$ then

$$\sum_{k=0}^{n-1} (2tM)^k/k! \leq (2M/\varepsilon)^{n-1} e^{\varepsilon t} \quad \text{for} \quad t \geq 0 .$$

Combining this with the inequality (4), we obtain the result.

It may be noted that, since $\alpha \geq -M$, if $\varepsilon \geq 2M$ then

$$|e^{tA}| \leq e^{(\alpha+\varepsilon)t} \quad \text{for} \quad t \geq 0 .$$

PROPOSITION 4. *Suppose* $A(t)$ *is a matrix function defined on an interval* J *such that*

$$|A(t_2) - A(t_1)| \leq \delta |t_2 - t_1| \quad \textit{for all} \quad t_1, t_2 \in J$$

and

$$|e^{tA(\tau)}| \leq Ke^{\alpha t} \quad \textit{for} \quad \tau \geq 0 , \; t \in J ,$$

where $\delta > 0$ *and* $K > 1$. *Then the fundamental matrix* $X(t)$ *of the equation* (2) *satisfies the inequality*

$$|X(t)X^{-1}(s)| \leq Ke^{\beta(t-s)} \quad \textit{for} \quad t \geq s ,$$

where $\beta = \alpha + (\delta K \log K)^{\frac{1}{2}}$.

Proof. For any fixed $u \in J$ we can write (2) in the form

$$x' = A(u)x + [A(t) - A(u)]x .$$

Hence, by the variation of constants formula, any solution $x(t)$ satisfies

$$x(t) = e^{(t-s)A(u)}x(s) + \int_s^t e^{(t-t')A(u)}[A(t') - A(u)]x(t')dt' .$$

Therefore, for $t \geq s$,

$$|x(t)| \leq Ke^{\alpha(t-s)}|x(s)| + K\int_s^t e^{\alpha(t-t')}|A(t') - A(u)||x(t')|dt' .$$

By Gronwall's inequality, this implies

$$|x(t)| \leq Ke^{\alpha(t-s)}|x(s)| \exp K\int_s^t |A(t') - A(u)|dt' \quad \text{for} \quad t \geq s .$$

Taking $u = (s + t)/2$ and using the Lipschitz condition, we get

$$|x(t)| \leq Ke^{\alpha(t-s)} e^{\delta K(t-s)^2/4}|x(s)| \quad \text{for} \quad t \geq s .$$

Put

$$h = 2\left(\frac{\log K}{\delta K}\right)^{\frac{1}{2}}, \quad \gamma = \tfrac{1}{2}(\delta K \log K)^{\frac{1}{2}} .$$

For $s \le t \le s + h$ we have

$$\delta K(t - s)/4 \le \delta K h/4 = \gamma$$

and hence

$$|x(t)| \le Ke^{(\alpha+\gamma)(t-s)}|x(s)| \quad \text{for} \quad s \le t \le s + h .$$

In general, if $s + nh \le t < s + (n + 1)h$ then

$$\begin{aligned} |x(t)| &\le Ke^{(\alpha+\gamma)(t-s-nh)}|x(s + nh)| \\ &\le \dots \\ &\le K^{n+1}e^{(\alpha+\gamma)(t-s)}|x(s)| . \end{aligned}$$

Since $\gamma = h^{-1} \log K$,

$$K^n = e^{n\gamma h} \le e^{\gamma(t-s)} .$$

Hence

$$|x(t)| \le Ke^{(\alpha+2\gamma)(t-s)}|x(s)| \quad \text{for all} \quad t \ge s .$$

By combining Propositions 3 and 4 we obtain at once

PROPOSITION 5. *Let* $A(t)$ *be an* $n \times n$ *matrix function defined on an interval* J *such that*

(i) *every eigenvalue of* $A(t)$ *has real part* $\le \alpha$ *for all* $t \in J$,

(ii) $|A(t)| \le M$ *for all* $t \in J$.

Then for any $\varepsilon > 0$ *there exists* $\delta = \delta(M, \varepsilon) > 0$ *such that if*

$$|A(t_2) - A(t_1)| \le \delta|t_2 - t_1| \quad \textit{for all} \quad t_1, t_2 \in J$$

the fundamental matrix $X(t)$ *of the equation* (2) *satisfies the inequality*

$$|X(t)X^{-1}(s)| \le K_\varepsilon e^{(\alpha+\varepsilon)(t-s)} \quad \textit{for} \quad t \ge s ,$$

where $K_\varepsilon = \max\{(4M/\varepsilon)^{n-1}, 1\}$.

In particular, (2) is uniformly asymptotically stable if $\alpha < 0$ and δ is sufficiently small.

Our next result shows that, if the coefficient matrices are bounded, uniform asymptotic stability is also preserved under 'integrally small' perturbations.

PROPOSITION 6. *Let* $A(t)$, $B(t)$ *be continuous matrix functions on an interval* *such that*

$$|A(t)| \le M \ , \ |B(t)| \le M \ ,$$

and suppose the fundamental matrix $X(t)$ *of the equation* (2) *satisfies the inequality*

$$|X(t)X^{-1}(s)| \le Ke^{\alpha(t-s)} \quad \textit{for} \quad t \ge s \ ,$$

where $K \ge 1$. *If*

$$\left|\int_{t_1}^{t_2} B(t)dt\right| \le \delta \quad \textit{for} \quad |t_2 - t_1| \le h$$

then the fundamental matrix $Y(t)$ *of the perturbed equation* (3) *satisfies the inequality*

$$|Y(t)Y^{-1}(s)| \le (1 + \delta)Ke^{\beta(t-s)} \quad \textit{for} \quad t \ge s \ ,$$

where $\beta = \alpha + 3MK\delta + h^{-1} \log(1 + \delta)K$.

Proof. By the variation of constants formula any solution $y(t)$ of (3) satisfies the integral equation

$$y(t) = X(t)X^{-1}(s)y(s) + \int_s^t X(t)X^{-1}(u)B(u)y(u)du \ .$$

If we set

$$C(s) = \int_s^t B(u)du$$

then, on integrating by parts and using (2) and (3), we obtain

$$y(t) = X(t)X^{-1}(s)[I + C(s)]y(s)$$
$$- \int_s^t X(t)X^{-1}(u)[A(u)C(u) - C(u)A(u) - C(u)B(u)]y(u)du \ .$$

It follows that for $s \le t \le s + h$

$$|y(t)| \le (1 + \delta)Ke^{\alpha(t-s)}|y(s)| + 3MK\delta\int_s^t e^{\alpha(t-u)}|y(u)|du \ .$$

Therefore, by Gronwall's inequality, $w(t) = e^{-\alpha t}|y(t)|$ satisfies

$$w(t) \le (1 + \delta)Ke^{3MK\delta(t-s)}w(s) \quad \text{for} \quad s \le t \le s + h \ .$$

For any s , $t \in J$ with $t \ge s$ there is a unique non-negative integer n such that $s + nh \le t < s + (n + 1)h$. Then

$$\begin{aligned} w(t) &\le (1 + \delta)Ke^{3MK\delta(t-s-nh)}w(s + nh) \\ &\le \dots \\ &\le [(1 + \delta)K]^{n+1}e^{3MK\delta(t-s)}w(s) \end{aligned}$$

and

$$|y(t)| \le [(1+\delta)K]^{n+1} e^{(\alpha+3MK\delta)(t-s)} |y(s)| .$$

If we put $\gamma = h^{-1}\log(1+\delta)K$ then

$$[(1+\delta)K]^n = e^{n\gamma h} \le e^{\gamma(t-s)} .$$

Hence

$$|y(t)| \le (1+\delta)Ke^{\beta(t-s)}|y(s)| \quad \text{for} \quad t \ge s ,$$

which is equivalent to the required inequality.

It is clear that, given any $\varepsilon > 0$, we will have $\beta < \alpha + \varepsilon$ if $h \ge 2\varepsilon^{-1}\log K$ and if $\delta = \delta(K, N, \varepsilon)$ is sufficiently small.

The integrally small perturbations of Proposition 6 form a more extensive class than the small perturbations of Proposition 1. Suppose, for example, that $\tilde{B}(t)$ is an almost periodic matrix function with mean value zero. We will show that for any $h > 0$, $\delta > 0$ there exists a corresponding $\omega_0 = \omega_0(h, \delta) > 0$ such that if $\omega > \omega_0$ then

$$\left|\int_{t_1}^{t_2} \tilde{B}(\omega t)dt\right| \le \delta \quad \text{for} \quad |t_2 - t_1| \le h .$$

In fact $\tilde{B}(t)$ is bounded, say $|\tilde{B}(t)| \le M$ for all t. Hence if $|t_2 - t_1| \le \delta/M$ then

$$\left|\int_{t_1}^{t_2} \tilde{B}(\omega t)dt\right| \le \delta \quad \text{for all} \quad \omega .$$

On the other hand, since $\tilde{B}(t)$ has mean value zero, there exists $T_0 = T_0(\delta/h) > 0$ such that if $T \ge T_0$ then

$$\left|T^{-1}\int_s^{s+T} \tilde{B}(t)dt\right| \le \delta/h \quad \text{for all} \quad s .$$

Therefore, if $\delta/M \le |t_2 - t_1| \le h$ and $\omega \ge \omega_0 = MT_0/\delta$,

$$\left|\int_{t_1}^{t_2} \tilde{B}(\omega t)dt\right| \le \left|\omega(t_2 - t_1)^{-1}\int_{\omega t_1}^{\omega t_2} \tilde{B}(u)du\right| h$$

$$\le \delta/h \cdot h = \delta .$$

Thus it follows from Proposition 6 that the uniform asymptotic stability of an autonomous system is not destroyed by sufficiently rapid oscillations. This is a counterpart to Proposition 5, which says that it is not destroyed by sufficiently slow variations.

We will say that the differential equation (2) has *bounded growth* on an interval J if, for some fixed $h > 0$, there exists a constant $C \ge 1$ such that every solution $x(t)$ of (2) satisfies

$$|x(t)| \le C|x(s)| \quad \text{for} \quad s\,,\, t \in J \quad \text{and} \quad s \le t \le s+h\,.$$

The equation (2) has bounded growth if and only if there exist real constants K , α such that its fundamental matrix $X(t)$ satisfies

$$|X(t)X^{-1}(s)| \le Ke^{\alpha(t-s)} \quad \text{for} \quad t \ge s\,. \tag{5}$$

In fact we can take $K = C$ and $\alpha = h^{-1} \log C$, resp. $C = Ke^{\alpha h}$. This shows that the condition of bounded growth is independent of the choice of h .

The equation (2) certainly has bounded growth if its coefficient matrix $A(t)$ is bounded, or even if $\int_t^{t+h}|A(s)|ds$ is bounded, since by Gronwall's inequality

$$|X(t)X^{-1}(s)| \le \exp\left|\int_s^t|A(u)|du\right|\,.$$

More generally, it has bounded growth if $\mu[A(t)]$ or $\int_t^{t+h}\mu[A(s)]ds$ is bounded above, where

$$\mu(A) = \lim_{\varepsilon\to+0}\{|I + \varepsilon A| - 1\}/\varepsilon$$

is the logarithmic norm of the matrix A , since

$$|X(t)X^{-1}(s)| \le \exp\int_s^t\mu[A(u)]du \quad \text{for} \quad t \ge s\,.$$

However, the preceding propositions provide smaller values for the exponent α of (5) in a number of cases.

In these lectures the hypothesis of bounded growth will often be imposed instead of the more restrictive hypothesis that the coefficient matrix is bounded. Actually, the concept of bounded growth is just the restriction to linear differential equations of the concept of uniform continuous dependence on initial conditions for a solution of an arbitrary differential equation.

Let $\tilde{x}(t)$ be a solution of the differential equation (1) on an interval J and let h be a fixed positive number. Then $\tilde{x}(t)$ may be said to be (right) - uniformly dependent on its initial value if for each $\varepsilon > 0$ there is a corresponding $\delta = \delta(\varepsilon) > 0$ such that any solution $x(t)$ of (1) which satisfies the inequality $|x(s) - \tilde{x}(s)| < \delta$ for some $s \in J$ is defined and satisfies the inequality $|x(t) - \tilde{x}(t)| < \varepsilon$ for all $t \in J$ with $s \le t \le s + h$.

2. EXPONENTIAL AND ORDINARY DICHOTOMIES

Let $X(t)$ be a fundamental matrix for the linear differential equation

$$x' = A(t)x \ , \tag{1}$$

where the $n \times n$ coefficient matrix $A(t)$ is continuous on an interval J. The equation (1) is said to possess an *exponential dichotomy* if there exists a projection P — that is, a matrix P such that $P^2 = P$ — and positive constants K, L, α, β such that

$$\begin{aligned} |X(t)PX^{-1}(s)| &\leq Ke^{-\alpha(t-s)} \quad \text{for} \quad t \geq s \ , \\ |X(t)(I - P)X^{-1}(s)| &\leq Le^{-\beta(s-t)} \quad \text{for} \quad s \geq t \ . \end{aligned} \tag{2}$$

It is said to possess an *ordinary dichotomy* if the inequalities (2) hold with α and β not positive but zero.

The two cases of most interest are where J is the positive half-line $\mathbb{R}_+$ and the whole line $\mathbb{R}$. *Until the final paragraph of this lecture we assume that* $J = \mathbb{R}_+$.

The autonomous equation

$$x' = A_0 x$$

has an exponential dichotomy on $\mathbb{R}_+$ if and only if no eigenvalue of the constant matrix A_0 has zero real part. It has an ordinary dichotomy if and only if all eigenvalues of A_0 with zero real part are semisimple. In each case $X(t) = e^{tA_0}$ and we can take P to be the spectral projection defined by

$$P = \frac{1}{2\pi i}\int_\gamma (zI - A_0)^{-1} dz \ ,$$

where γ is any rectifiable simple closed curve in the open left half-plane which contains in its interior all eigenvalues of A_0 with negative real part. The concepts of exponential and ordinary dichotomy generalize to non-autonomous equations these two types of conditional stability.

The equation (1) has an exponential dichotomy with $P = I$ if and only if it is uniformly asymptotically stable, and an ordinary dichotomy with $P = I$ if and only if it is uniformly stable. To see what a dichotomy means in the general case it is convenient to rewrite the conditions (2) in the equivalent form

$$\begin{gathered} |X(t)P\xi| \le K'e^{-\alpha(t-s)}|X(s)P\xi| \quad \text{for} \quad t \ge s , \\ |X(t)(I - P)\xi| \le L'e^{-\beta(s-t)}|X(s)(I - P)\xi| \quad \text{for} \quad s \ge t , \\ |X(t)PX^{-1}(t)| \le M' \quad \text{for all} \quad t , \end{gathered} \tag{2'}$$

where K', L', M' are again positive constants and ξ is an arbitrary constant vector. Suppose the projection P has rank k, i.e. trace $P = k$, and for definiteness assume α, $\beta > 0$. Then the first condition (2)' says that there is a k-dimensional subspace of solutions tending to zero uniformly and exponentially as $t \to \infty$. The second condition says that there is a supplementary $(n - k)$-dimensional subspace of solutions tending to infinity uniformly and exponentially as $t \to \infty$. The third condition says that the 'angle' between these two subspaces remains bounded away from zero.

If α or β is positive and the equation (1) has bounded growth then the third condition (2)' is actually implied by the previous two conditions. For suppose

$$|X(t)X^{-1}(s)| \le Ce^{\mu(t-s)} \quad \text{for} \quad t \ge s , \tag{3}$$

where $C \ge 1$ and $\mu > 0$ are constants. For any $h > 0$ we have

$$|X(t + h)PX^{-1}(t)| \le K'e^{-\alpha h}|X(t)PX^{-1}(t)| ,$$

$$|X(t + h)(I - P)X^{-1}(t)| \ge L'^{-1}e^{\beta h}|X(t)(I - P)X^{-1}(t)| .$$

Putting

$$\rho = |X(t)(I - P)X^{-1}(t)| , \quad \sigma = |X(t)PX^{-1}(t)| ,$$

we obtain

$$|\rho^{-1}X(t + h)(I - P)X^{-1}(t) + \sigma^{-1}X(t + h)PX^{-1}(t)| \ge \gamma ,$$

where $\gamma = L'^{-1}e^{\beta h} - K'e^{-\alpha h}$. Choose $h > 0$ so large that $\gamma > 0$. Then, by (3),

$$|\rho^{-1}X(t)(I - P)X^{-1}(t) + \sigma^{-1}X(t)PX^{-1}(t)| \ge \gamma C^{-1}e^{-\mu h} .$$

The left side of this inequality can be written in the form

$$|\sigma^{-1}I + (\rho^{-1} - \sigma^{-1})X(t)(I - P)X^{-1}(t)| \le \sigma^{-1} + |\rho^{-1} - \sigma^{-1}|\rho$$
$$= \sigma^{-1}\{1 + |\sigma - \rho|\}$$
$$\le 2\sigma^{-1} .$$

Hence $\sigma \le 2\gamma^{-1}Ce^{\mu h}$ and, by symmetry, also $\rho \le 2\gamma^{-1}Ce^{\mu h}$. It is readily shown that if $P \neq 0$, I we can choose $h > 0$ so that

$$\rho \ , \quad \sigma \le 2eC\frac{\mu+\alpha}{\beta+\alpha} L'(K'L')^{(\mu-\beta)/(\beta+\alpha)} .$$

If α and β are positive but the equation (1) does not have bounded growth then the third condition (2)' is not implied by the previous two conditions. For example, the second order system

$$x_1' = -x_1 + e^{2t}x_2$$
$$x_2' = x_2$$

has the fundamental matrix

$$X(t) = \begin{pmatrix} e^{-t} & (e^{3t} - e^{-t})/4 \\ 0 & e^{t} \end{pmatrix} .$$

It may be verified without difficulty that if

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

then the first two conditions (2)' are satisfied, with $\alpha = 1$ and $\beta = 3$, but the third condition is not satisfied.

If $\alpha = \beta = 0$ then, even if the coefficient matrix $A(t)$ is bounded, the third condition (2)' is not implied by the previous two conditions. For example, the second order system

$$x_1' = x_2$$
$$x_2' = 0$$

has the fundamental matrix

$$X(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} .$$

If

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

then the first two conditions (2)' are satisfied, with $\alpha = \beta = 0$, but the third condition is not satisfied.

If the equation (1) has an exponential or ordinary dichotomy (2) on a sub-interval $[t_0, \infty)$ then it also has an exponential or ordinary dichotomy on the half-line $\mathbb{R}_+$, with the same projection P and the same exponents α, β. In fact we can choose $N \geq 1$, for example

$$N = \exp \int_0^{t_0} |A(u)| du ,$$

so that

$$|X(t)X^{-1}(s)| \leq N \quad \text{for} \quad 0 \leq s , t \leq t_0 .$$

If $0 \leq s \leq t_0 \leq t$ then

$$\begin{aligned} |X(t)PX^{-1}(s)| &\leq N|X(t)PX^{-1}(t_0)| \\ &\leq NKe^{-\alpha(t-t_0)} \\ &\leq NKe^{\alpha t_0} e^{-\alpha(t-s)} . \end{aligned}$$

If $0 \leq s \leq t \leq t_0$ then

$$\begin{aligned} |X(t)PX^{-1}(s)| &\leq N^2|X(t_0)PX^{-1}(t_0)| \\ &\leq N^2K \\ &\leq N^2Ke^{\alpha t_0} e^{-\alpha(t-s)} . \end{aligned}$$

Hence

$$|X(t)PX^{-1}(s)| \leq \tilde{K}e^{-\alpha(t-s)} \quad \text{for} \quad 0 \leq s \leq t < \infty ,$$

where $\tilde{K} = N^2Ke^{\alpha t_0}$. Similarly,

$$|X(t)(I - P)X^{-1}(s)| \leq \tilde{L}e^{-\beta(s-t)} \quad \text{for} \quad 0 \leq t \leq s < \infty ,$$

where $\tilde{L} = N^2Le^{\beta t_0}$.

Suppose the equation (1) satisfies the first two conditions (2)' with α, $\beta > 0$. For any solution $x(t)$ of (1) write

$$x_1(t) = X(t)PX^{-1}(t)x(t) \ , \ x_2(t) = X(t)(I - P)X^{-1}(t)x(t) \ ,$$

so that

$$x(t) = X(t)PX^{-1}(s)x_1(s) + X(t)(I - P)X^{-1}(s)x_2(s) \ .$$

If $|x_2(s)| \geq |x_1(s)|$ then, for $t \geq s$,

$$|x(t)| \geq L'^{-1}e^{\beta(t-s)}|x_2(s)| - K'e^{-\alpha(t-s)}|x_1(s)|$$

$$\geq \{L'^{-1}e^{\beta(t-s)} - K'e^{-\alpha(t-s)}\}|x_2(s)|$$

$$\geq \tfrac{1}{2}\{L'^{-1}e^{\beta(t-s)} - K'e^{-\alpha(t-s)}\}|x(s)| \ .$$

Similarly, if $|x_2(s)| \leq |x_1(s)|$ then, for $t \leq s$,

$$|x(t)| \geq \tfrac{1}{2}\{K'^{-1}e^{\alpha(s-t)} - L'e^{-\beta(s-t)}\}|x(s)| \ .$$

For any θ such that $0 < \theta < 1$ we can choose $T > 0$ so large that

$$L'^{-1}e^{\beta T} - K'e^{-\alpha T} \geq 2\theta^{-1} \ , \ K'^{-1}e^{\alpha T} - L'e^{-\beta T} \geq 2\theta^{-1} \ .$$

Then in either case we will have

$$|x(s)| \leq \theta \sup_{|u-s|\leq T} |x(u)| \quad \text{for every} \quad s \geq T \ .$$

We are going to show that if the equation (1) has bounded growth then this necessary condition for an exponential dichotomy is also sufficient.

PROPOSITION 1. *The equation* (1) *has an exponential dichotomy on* $\mathbb{R}_+$ *if there exist constants* $T > 0$, $C > 1$ *and* $0 < \theta < 1$ *such that every solution* $x(t)$ *of* (1) *satisfies*

$$|x(t)| \leq C|x(s)| \quad \textit{for} \quad 0 \leq s \leq t \leq s + T$$

and

$$|x(t)| \leq \theta \sup_{|u-t|\leq T} |x(u)| \quad \textit{for} \quad t \geq T \ . \tag{4}$$

Proof. Suppose first that $x(t)$ is a nontrivial bounded solution and for any $s \geq 0$ set

$$\mu(s) = \sup_{u\geq s} |x(u)| \ .$$

Then for $t \geq s + T$

$$|x(t)| \leq \theta \sup_{|u-t|\leq T} |x(u)| \leq \theta\mu(s)$$

and hence

$$\mu(s) = \sup_{s\le u\le s+T} |x(u)| .$$

It follows that

$$|x(t)| \le C|x(s)| \quad \text{for} \quad 0 \le s \le t < \infty .$$

If $s + nT \le t < s + (n + 1)T$ then

$$\begin{aligned} |x(t)| &\le \theta^n \sup_{|u-t|\le nT} |x(u)| \\ &\le \theta^n C|x(s)| \\ &\le \theta^{-1}C\theta^{(t-s)/T}|x(s)| . \end{aligned}$$

Thus

$$|x(t)| \le Ke^{-\alpha(t-s)}|x(s)| \quad \text{for} \quad 0 \le s \le t < \infty ,$$

where $K = \theta^{-1}C > 1$ and $\alpha = -T^{-1} \log \theta > 0$.

Suppose next that $x(t)$ is an unbounded solution with $|x(0)| = 1$. We can define $t_n > 0$ by

$$|x(t_n)| = \theta^{-n}C , \; |x(t)| < \theta^{-n}C \quad \text{for} \quad 0 \le t < t_n .$$

Then $T < t_1 < t_2 < \ldots$ and $t_n \to \infty$. Moreover $t_{n+1} \le t_n + T$, since

$$|x(t_n)| \le \theta \sup_{0\le u\le t_n+T} |x(u)|$$

and $|x(u)| < \theta^{-1}|x(t_n)|$ for $0 \le u < t_{n+1}$. Suppose $t \le s$ and $t_m \le t < t_{m+1}$, $t_n \le s < t_{n+1}$ $(1 \le m \le n)$. Then

$$\begin{aligned} |x(t)| < \theta^{-m-1}C &= \theta^{n-m}|x(t_{n+1})| \\ &\le C\theta^{-1}\theta^{n-m+1}|x(s)| \\ &\le C\theta^{-1}\theta^{(s-t)/T}|x(s)| . \end{aligned}$$

Thus

$$|x(t)| \le Ke^{-\alpha(s-t)}|x(s)| \quad \text{for} \quad t_1 \le t \le s < \infty .$$

Let V be the underlying vector space ($\mathbb{R}^n$ or $\mathbb{C}^n$) , let V_1 be the subspace consisting of the initial values of all bounded solutions of (1), and let V_2 be any fixed subspace supplementary to V_1 . For any unit vector $\xi \in V_2$, let $x(t) = x(t, \xi)$ be the solution which takes the value ξ at $t = 0$. Then $x(t, \xi)$ is unbounded, and hence there is a least value $t_1 = t_1(\xi)$ such that $|x(t_1, \xi)| = \theta^{-1}C$. We will show that the values $t_1(\xi)$ are bounded. In fact, otherwise there exists a sequence of unit vectors $\xi_\nu \in V_2$ such that $t_1^{(\nu)} = t_1(\xi_\nu) \to \infty$. By the compactness of the unit sphere in V_2 we may suppose that $\xi_\nu \to \xi$, where $|\xi| = 1$. Then

$$x(t, \xi_\nu) \to x(t, \xi) \quad \text{for every} \quad t \geq 0 .$$

Since $|x(t, \xi_\nu)| < \theta^{-1}C$ for $0 \leq t < t_1^{(\nu)}$ it follows that

$$|x(t, \xi)| \leq \theta^{-1}C \quad \text{for} \quad 0 \leq t < \infty ,$$

which is a contradiction because $\xi \in V_2$.

Thus there exists $T_1 > 0$ such that $t_1(\xi) \leq T_1$ for all ξ , and every solution $x(t)$ with $x(0) \in V_2$ satisfies

$$|x(t)| \leq Ke^{-\alpha(s-t)}|x(s)| \quad \text{for} \quad T_1 \leq t \leq s < \infty .$$

Consequently the first two conditions (2)' are satisfied on the subinterval $[T_1, \infty)$. The third condition (2)' is redundant, since (1) has bounded growth. Therefore (1) has an exponential dichotomy on $[T_1, \infty)$, and hence also on $\mathbb{R}_+$.

We consider next the possibility of varying the projection in a dichotomy for a given fundamental matrix. Suppose the equation (1) possesses an exponential or ordinary dichotomy (2) with projection P , corresponding to the fundamental matrix $X(t)$ with $X(0) = I$. We will show that if P' is a projection with the same range as P then the equation (1) also possesses an exponential or ordinary dichotomy with projection P' .

We have

$$P'P = P \ , \ PP' = P' \ ,$$

and hence

$$P - P' = P(P - P') = (P - P')(I - P) .$$

Therefore, for any vector ξ and all s , $t \geq 0$

$$|X(t)(P - P')\xi| \le Ke^{-\alpha t}|(P - P')\xi|$$
$$\le Ke^{-\alpha t}|P' - P||(I - P)\xi|$$
$$\le KLe^{-\alpha t}e^{-\beta s}|P' - P||X(s)\xi| \ .$$

It follows that for $0 \le t \le s$

$$|X(t)(I - P')\xi| \le |X(t)(I - P)\xi| + |X(t)(P - P')\xi|$$
$$\le \{1 + K|P' - P|\}Le^{-\beta(s-t)}|X(s)\xi| \ .$$

Similarly, for $0 \le s \le t$

$$|X(t)P'\xi| \le \{1 + L|P' - P|\}Ke^{-\alpha(t-s)}|X(s)\xi| \ .$$

Therefore (1) has a dichotomy with projection P' , the exponents α , β being unaltered and the constants K , L being multiplied by $1 + L|P' - P|$, $1 + K|P' - P|$ respectively.

In the case of an exponential dichotomy this is the only indeterminacy in the choice of the projection P since, if $X(0) = I$, the range of P is uniquely determined as the subspace consisting of the initial values of all bounded solutions.

However, in the case of an ordinary dichotomy there may be other choices for the projection P . Let V be the underlying vector space, let V_1 be the subspace of V consisting of the initial values of all bounded solutions of (1), and let V_0 be the subspace of V_1 consisting of the initial values of all solutions of (1) which tend to zero as $t \to \infty$. Then

PROPOSITION 2. *Let* $X(t)$ *be the fundamental matrix of* (1) *with* $X(0) = I$. *If the equation* (1) *has a dichotomy* (2) *with projection* P *then it also has a dichotomy with projection* Q *if and only if*

$$V_0 \subseteq QV \subseteq V_1 \ . \tag{5}$$

Proof. We may assume that (2) holds with $\alpha = \beta = 0$ and $K = L$. Let $x(t)$ be a nontrivial solution of (1) such that $|x(t_n)| \to 0$ for some sequence of numbers $t_n \ge 0$. Since $x(t)$ is nontrivial we must have $t_n \to \infty$. Put

$$x_1(t) = X(t)Px(0) \ , \ x_2(t) = X(t)(I - P)x(0) \ .$$

Then $x(t) = x_1(t) + x_2(t)$ and

$$|x_1(t)| \le K|x(t_n)| \quad \text{for} \quad t \ge t_n \ ,$$
$$|x_2(t)| \le K|x(t_n)| \quad \text{for} \quad 0 \le t \le t_n \ .$$

The first relation implies that $|x_1(t)| \to 0$ as $t \to \infty$ and the second implies that $x_2(t) = 0$ for all $t \geq 0$. Hence $x(0) \in P\mathcal{V}$ and $|x(t)| \to 0$ as $t \to \infty$.

Thus $\mathcal{V}_0 \subseteq P\mathcal{V}$. On the other hand $P\mathcal{V} \subseteq \mathcal{V}_1$, since $|X(t)P| \leq K$ for $t \geq 0$. This proves the necessity of the condition (5).

Let P_0 be a projection with range $\mathcal{V}_0$ whose nullspace contains the nullspace of P. Then $\mathcal{V}'_0 = (P - P_0)\mathcal{V}$ is a subspace of $P\mathcal{V}$ supplementary to $\mathcal{V}_0$. We will show that there exists a constant $N > 0$ such that

$$|X(t)\xi| \geq N|\xi| \quad \text{for } \xi \in \mathcal{V}'_0 \text{ and } t \geq 0 .$$

In fact, otherwise there exists a sequence of vectors $\xi_\nu \in \mathcal{V}'_0$ with $|\xi_\nu| = 1$ and a sequence of numbers $t_\nu \geq 0$ such that $|X(t_\nu)\xi_\nu| \to 0$. By restricting attention to a subsequence we may assume that $\xi_\nu \to \xi$, where $\xi \in \mathcal{V}'_0$ and $|\xi| = 1$. Since

$$\begin{aligned} |X(t_\nu)\xi| &\leq |X(t_\nu)\xi_\nu| + |X(t_\nu)(\xi - \xi_\nu)| \\ &\leq |X(t_\nu)\xi_\nu| + K|\xi - \xi_\nu| \end{aligned}$$

it follows that $|X(t_\nu)\xi| \to 0$. Therefore $\xi \in \mathcal{V}_0$, which is a contradiction.

Thus for any vector $\xi \in \mathcal{V}$ we have, for $0 \leq s \leq t$,

$$\begin{aligned} N|(P - P_0)\xi| &\leq |X(t)(P - P_0)\xi| \\ &\leq |X(t)P\xi| + |X(t)P_0\xi| \\ &\leq K|X(s)\xi| + |X(t)P_0\xi| . \end{aligned}$$

Letting $t \to \infty$ we obtain

$$N|(P - P_0)\xi| \leq K|X(s)\xi| \quad \text{for all } s \geq 0 .$$

Therefore, for any vector $\xi \in \mathcal{V}$ and all $s, t \geq 0$

$$\begin{aligned} |X(t)(P - P_0)\xi| &\leq K|(P - P_0)\xi| \\ &\leq N^{-1}K^2|X(s)\xi| . \end{aligned}$$

Hence

$$\begin{aligned} |X(t)P_0\xi| &\leq |X(t)P\xi| + |X(t)(P - P_0)\xi| \\ &\leq (1 + N^{-1}K)K|X(s)\xi| \quad \text{for } 0 \leq s \leq t \end{aligned}$$

and

$$|X(t)(I - P_0)\xi| \le |X(t)(I - P)\xi| + |X(t)(P - P_0)\xi|$$

$$\le (1 + N^{-1}K)K|X(s)\xi| \quad \text{for} \quad 0 \le t \le s .$$

Thus (1) has a dichotomy with projection P_0 .

Suppose now that Q is a projection satisfying the condition (5). By what we have just proved we may assume that $PV \subseteq QV$, i.e., $QP = P$. There exists a constant $N' > 0$ such that

$$|X(t)\xi| \le N'|\xi| \quad \text{for} \quad \xi \in V_1 \quad \text{and} \quad t \ge 0 .$$

Therefore, for any vector $\xi \in V$ and all $s , t \ge 0$

$$\begin{aligned} |X(t)(Q - P)\xi| &\le N'|(Q - P)\xi| \\ &\le N'|Q||(I - P)\xi| \\ &\le KN'|Q||X(s)\xi| . \end{aligned}$$

It follows that for $0 \le s \le t$

$$|X(t)Q\xi| \le (1 + N'|Q|)K|X(s)\xi|$$

and for $0 \le t \le s$

$$|X(t)(I - Q)\xi| \le (1 + N'|Q|)K|X(s)\xi| .$$

Thus (1) has a dichotomy with projection Q . This completes the proof.

Finally we remark that in the case of an exponential dichotomy on the whole line $\mathbb{R}$ the projection P is *uniquely* determined. In fact, if $X(0) = I$, the range of P is the subspace of initial values of solutions bounded on the positive half-line $\mathbb{R}_+$ and the nullspace of P is the subspace of initial values of solutions bounded on the negative half-line $\mathbb{R}_-$. Also, if the equation (1) has an exponential, or ordinary, dichotomy (2) on each of the half-lines $\mathbb{R}_+$, $\mathbb{R}_-$ with the same projection P then it has an exponential, or ordinary, dichotomy on the whole line $\mathbb{R}$ with projection P .

3. DICHOTOMIES AND FUNCTIONAL ANALYSIS

Throughout this lecture we will assume that $A(t)$ is a continuous matrix function defined on the half-line $\mathbb{R}_+$ and we will denote by $X(t)$ the fundamental matrix of the linear differential equation

$$x' = A(t)x \tag{1}$$

such that $X(0) = I$. A vector function $f(t)$ will be said to be *locally integrable* if it is measurable and $\int_J |f(t)|dt < \infty$ for every compact interval $J \subset \mathbb{R}_+$. If $f(t)$ is locally integrable then by a solution of the inhomogeneous equation

$$y' = A(t)y + f(t) \tag{2}$$

we will mean an absolutely continuous function $y(t)$ which satisfies (2) for almost all t.

Suppose first that (1) has an exponential dichotomy on $\mathbb{R}_+$. Thus there exists a projection P and positive constants K, α such that

$$\begin{aligned} |X(t)PX^{-1}(s)| &\le Ke^{-\alpha(t-s)} \quad \text{for } 0 \le s \le t , \\ |X(t)(I-P)X^{-1}(s)| &\le Ke^{-\alpha(s-t)} \quad \text{for } 0 \le t \le s . \end{aligned} \tag{3}$$

Then for any bounded continuous function $f(t)$ the corresponding inhomogeneous equation (2) has a bounded solution. In fact, it is easily verified that

$$y(t) = \int_0^t X(t)PX^{-1}(s)f(s)ds - \int_t^\infty X(t)(I-P)X^{-1}(s)f(s)ds \tag{4}$$

is a solution of (2) and

$$\sup_{t\ge0} |y(t)| \le 2\alpha^{-1}K \sup_{t\ge0} |f(t)| .$$

Actually the formula (4) defines a bounded solution of (2) not only for any bounded continuous function $f(t)$ but also for any locally integrable function $f(t)$ such that $\int_t^{t+1}|f(s)|ds$ is bounded. This follows from

LEMMA 1. *Let* $\gamma(t)$ *be a non-negative locally integrable function such that*

$$\int_t^{t+1}\gamma(s)ds \le C \quad \textit{for all} \quad t \ge 0 .$$

If $\alpha > 0$ *then, for all* $t \ge 0$,

$$\int_0^t e^{-\alpha(t-s)}\gamma(s)ds \le (1 - e^{-\alpha})^{-1}C ,$$

$$\int_t^\infty e^{-\alpha(s-t)}\gamma(s)ds \le (1 - e^{-\alpha})^{-1}C .$$

Proof. From

$$\int_{t-m-1}^{t-m} e^{-\alpha(t-s)}\gamma(s)ds \le e^{-\alpha t}e^{\alpha(t-m)}C = e^{-\alpha m}C$$

we obtain

$$\int_0^t e^{-\alpha(t-s)}\gamma(s)ds \le \sum_{m=0}^{\infty} e^{-\alpha m}C = (1 - e^{-\alpha})^{-1}C .$$

The second inequality is proved similarly.

Suppose next that (1) has an ordinary dichotomy on $\mathbb{R}_+$. Thus (3) holds with $\alpha = 0$. Then for any locally integrable function $f(t)$ with $\int_0^\infty|f(t)|dt < \infty$ the corresponding inhomogeneous equation (2) has a bounded solution, defined by the same formula (4).

We intend to establish converses to these results. Let $\mathbf{C}$ denote the Banach space of all bounded continuous vector functions f , with the norm

$$\|f\|_C = \sup_{t\ge0} |f(t)| .$$

Let $\mathbf{M}$ denote the Banach space of all locally integrable vector functions f for which $\int_t^{t+1}|f(s)|ds$ is bounded, with the norm

$$\|f\|_M = \sup_{t\ge0}\int_t^{t+1}|f(s)|ds .$$

(The same space $\mathbf{M}$, with an equivalent norm, would have been obtained if instead of intervals of length 1 we had used intervals of any fixed length $h > 0$). Finally, let $\mathbf{L}$ denote the Banach space of all vector functions f which are Lebesgue integrable on $\mathbb{R}_+$, with the norm

$$\|f\|_L = \int_0^\infty|f(t)|dt .$$

Then we will prove

PROPOSITION 1. *The inhomogeneous equation* (2) *has at least one bounded solution for every function* $f \in L$ *if and only if the homogeneous equation* (1) *has an ordinary dichotomy.*

PROPOSITION 2. *The inhomogeneous equation* (2) *has at least one bounded solution for every function* $f \in M$ *if and only if the homogeneous equation* (1) *has an exponential dichotomy.*

PROPOSITION 3. *Suppose* (1) *has bounded growth. Then the inhomogeneous equation* (2) *has at least one bounded solution for every function* $f \in C$ *if and only if the homogeneous equation* (1) *has an exponential dichotomy.*

Let V be the underlying vector space ($\mathbb{R}^n$ or $\mathbb{C}^n$), let V_1 be the subspace of V consisting of the initial values of all bounded solutions of (1), and let V_2 be any fixed subspace of V supplementary to V_1. Also let P denote the projection with range V_1 and nullspace V_2. The basic result is

PROPOSITION 4. *Suppose the equation* (2) *has a bounded solution for every function* $f \in B$, *where* B *denotes any one of the Banach spaces* L, M, C. *Then there exists a least constant* $r_B = r_B(P) > 0$ *such that, for every* $f \in B$, *the unique bounded solution* $y(t)$ *of* (2) *with* $y(0) \in V_2$ *satisfies*

$$\|y\|_C \leq r_B \|f\|_B \ .$$

Proof. It follows from the superposition principle that, for every $f \in B$, the equation (2) does have a unique bounded solution $y(t)$ with $y(0) \in V_2$. Moreover the map $T : f \to y$ is linear. We will show that T has a closed graph. Suppose $f_n \to f$ in B and $y_n = Tf_n \to y$ in C. Then

$$y(0) = \lim_{n\to\infty} y_n(0) \in V_2$$

and, for any fixed t,

$$\int_0^t f(s)ds = \lim_{n\to\infty} \int_0^t f_n(s)ds \ .$$

Hence

$$y(t) - y(0) = \lim_{n\to\infty} \int_0^t y_n'(s)ds$$

$$= \lim_{n\to\infty} \int_0^t \{A(s)y_n(s) + f_n(s)\}ds$$

$$= \int_0^t \{A(s)y(s) + f(s)\}ds \ .$$

Thus $y(t)$ is a solution of the equation (2). Therefore, since it is bounded, $y = Tf$.

It now follows from the Closed Graph Theorem that the linear map T is continuous. This proves the existence of some constant $r_B > 0$, and the existence of a least one can be deduced immediately.

Put

$$G(t, s) = \begin{cases} X(t)PX^{-1}(s) & \text{for } 0 \le s < t \ , \\ -X(t)(I - P)X^{-1}(s) & \text{for } 0 \le t < s \ . \end{cases}$$

Then G is continuous except on the line $s = t$, where it has a jump discontinuity. If f is a function in B which vanishes for $t > t_1$ then — cf. (4) —

$$y(t) = \int_0^{t_1} G(t, u)f(u)du$$

is a solution of (2). Moreover it is bounded, since

$$y(t) = X(t)P\int_0^{t_1} X^{-1}(u)f(u)du \quad \text{for} \quad t \ge t_1 \ ,$$

and

$$y(0) = -(I - P)\int_0^{t_1} X^{-1}(u)f(u)du \in V_2 \ .$$

Therefore, by Proposition 4,

$$\|y\|_C \le r_B \|f\|_B \ .$$

We can now complete the proof of Proposition 1. Let ξ be any constant vector and let f be the function defined by

$$f(t) = \begin{cases} \xi & \text{for } s \le t \le s + h \ , \\ 0 & \text{otherwise}, \end{cases}$$

where $s \ge 0$ and $h > 0$. Then $f \in L$ and $\|f\|_L = h|\xi|$. Therefore

$$|y(t)| = \left|\int_s^{s+h} G(t, u)\xi du\right| \le r_L h|\xi| \ .$$

Dividing by h and letting $h \to 0$, we obtain for any $t \ne s$

$$|G(t, s)\xi| \le r_L|\xi| \ .$$

Hence, since ξ is arbitrary,

$$|G(t, s)| \le r_L .$$

Thus (3) holds with $K = r_L$ and $\alpha = 0$. By continuity, (3) remains valid also in the excepted case $s = t$.

The deduction of Propositions 2 and 3 is not quite so immediate. We suppose now that the equation (2) has a bounded solution for every function $f \in C$ and put $r = r_C$. Take

$$f(t) = \phi(t)x(t)/|x(t)|$$

where $x(t) = X(t)\xi$ is any nontrivial solution of the homogeneous equation (1) and $\phi(t)$ is any continuous real-valued function such that $0 \le \phi(t) \le 1$ for all $t \ge 0$, $\phi(t) = 0$ for $t \ge t_1$. Then $\|f\|_C \le 1$ and hence, by the arbitrary nature of ϕ,

$$\left|\int_{t_0}^{t_1} G(t, u)x(u)|x(u)|^{-1}du\right| \le r \quad \text{for} \quad 0 \le t_0 \le t_1 \quad \text{and} \quad t \ge 0 .$$

Putting $t_1 = t$, resp. $t_0 = t$, we obtain

$$\begin{aligned} |X(t)P\xi|\int_{t_0}^{t} |X(u)\xi|^{-1}du &\le r \quad \text{for} \quad t \ge t_0 \ge 0 , \\ |X(t)(I - P)\xi|\int_{t}^{t_1}|X(u)\xi|^{-1}du &\le r \quad \text{for} \quad t \le t_1 \le \infty . \end{aligned} \tag{5}$$

Replacing ξ by $P\xi$, resp. $(I - P)\xi$, it follows by integration that

$$\begin{aligned} \int_{t_0}^{s} |X(u)P\xi|^{-1}du &\le e^{-r^{-1}(t-s)}\int_{t_0}^{t} |X(u)P\xi|^{-1}du \quad \text{for} \quad t \ge s \ge t_0 , \\ \int_{s}^{t_1}|X(u)(I - P)\xi|^{-1}du &\le e^{-r^{-1}(s-t)}\int_{t}^{t_1}|X(u)(I - P)\xi|^{-1}du \quad \text{for} \quad t \le s \le t_1 . \end{aligned} \tag{6}$$

We use these inequalities to establish

LEMMA 2. *Suppose the equation* (2) *has a bounded solution for every function* $f \in C$ *and let* $r = r_C$. *Let* $x(t)$ *be a solution of the corresponding homogeneous equation* (1) *and put*

$$x_1(t) = X(t)PX^{-1}(t)x(t) , \quad x_2(t) = X(t)(I - P)X^{-1}(t)x(t) .$$

If, for some fixed $s \ge 0$,

$$|x_1(t)| \le N|x(s)| \quad \textit{for} \quad s \le t \le s + r$$

then

$$|x_1(t)| \le eN|x(s)|e^{-r^{-1}(t-s)} \quad \textit{for} \quad s \le t < \infty .$$

If, for some fixed $s \ge 0$,

$$|x_2(t)| \le N|x(s)| \quad \textit{for} \quad \max(0 , s - r) \le t \le s$$

then

$$|x_2(t)| \le eN|x(s)|e^{-r^{-1}(s-t)} \quad \textit{for} \quad 0 \le t \le s .$$

Proof. We can write $x(t) = X(t)\xi$ for some vector ξ . Replacing t_0 by s and s by $s + r$ in the first inequality (6) we obtain for $t \ge s + r$

$$r/N|x(s)| \le \int_s^{s+r}|x_1(u)|^{-1}du$$

$$\le e\, e^{-r^{-1}(t-s)}\int_s^t|x_1(u)|^{-1}du .$$

Using the first inequality (5), this gives for $t \ge s + r$

$$|x_1(t)| \le r\left[\int_s^t|x_1(u)|^{-1}du\right]^{-1}$$

$$\le eN|x(s)|e^{-r^{-1}(t-s)} .$$

Since the same inequality evidently holds also for $s \le t \le s + r$, this proves the first assertion of the lemma.

The proof of the second assertion is similar, replacing s by $s - r$ and t_1 by s in the second inequality (6).

We now conclude the proof of Proposition 2. Since $C \subset M$ and $L \subset M$, the equation (2) has a bounded solution for every $f \in C$ and for every $f \in L$. This is all the information that we will actually use. By Proposition 1 and its proof, (3) holds with $K = r_L$ and $\alpha = 0$. Hence in Lemma 2 we can take $N = r_L$, for every solution $x(t)$ of (1) and every $s \ge 0$, and obtain (3) with $K = er_L$ and $\alpha = r_C^{-1}$.

We turn next to Proposition 3 and assume that (1) has bounded growth. Thus

$$|X(t)X^{-1}(s)| \le Ce^{\mu(t-s)} \quad \text{for} \quad t \ge s ,$$

where $C \ge 1$ and $\mu > 0$ are constants. Replacing ξ by $X^{-1}(s)\xi$ and putting $t_1 = \infty$ in the second inequality (5), this gives for $t \le s$

$$|X(t)(I - P)X^{-1}(s)\xi| \le r\left[\int_s^\infty |X(u)X^{-1}(s)\xi|^{-1}du\right]^{-1}$$

$$\le r\left[C^{-1}|\xi|^{-1}\int_s^\infty e^{\mu(s-u)}du\right]^{-1} .$$

Thus

$$|X(t)(I - P)X^{-1}(s)| \le \mu rC \quad \text{for} \quad t \le s .$$

In the same way

$$|X(t)(I - P)X^{-1}(s)| \le \mu rCe^{\mu(t-s)} \quad \text{for} \quad t \ge s ,$$

and hence

$$|X(t)PX^{-1}(s)| \le (1 + \mu r)Ce^{\mu(t-s)} \quad \text{for} \quad t \ge s .$$

Similarly, from the first inequality (5) we obtain

$$|X(t)PX^{-1}(s)| \le \mu rC\left[1 - e^{-\mu(t-s)}\right]^{-1} \quad \text{for} \quad t > s .$$

By using this inequality for $t - s \ge h$, where

$$h = \mu^{-1} \log\left(\frac{1+2\mu r}{1+\mu r}\right) ,$$

and the previous inequality for $t - s \le h$, we get

$$|X(t)PX^{-1}(s)| \le (1 + 2\mu r)C \quad \text{for all} \quad t \ge s .$$

It now follows from Lemma 2 that

$$|X(t)PX^{-1}(s)| \le e(1 + 2\mu r)Ce^{-r^{-1}(t-s)} \quad \text{for} \quad 0 \le s \le t ,$$

$$|X(t)(I - P)X^{-1}(s)| \le e\mu rCe^{-r^{-1}(s-t)} \quad \text{for} \quad 0 \le t \le s .$$

Thus (1) has an exponential dichotomy.

When the coefficient matrix $A(t)$ is bounded, which implies that (1) has bounded growth, we can obtain slightly sharper values for the constants of the exponential dichotomy. We will show that if $|A(t)| \le M$ for $t \ge 0$ then

$$|X(t)PX^{-1}(s)| \le e(1 + rM)e^{-r^{-1}(t-s)} \quad \text{for} \quad 0 \le s \le t ,$$

$$|X(t)(I - P)X^{-1}(s)| \le erMe^{-r^{-1}(s-t)} \quad \text{for} \quad 0 \le t \le s .$$

In fact, for any unit vector ξ put

$$y(t) = \begin{cases} -X(t)(I - P)X^{-1}(s)\xi & \text{for } 0 \le t \le s , \\ X(t)PX^{-1}(s)\xi - \xi & \text{for } t \ge s . \end{cases}$$

Then $y(0) \in V_2$ and $y(t)$ is a bounded solution of the inhomogeneous equation (2) with

$$f(t) = \begin{cases} 0 & \text{for } 0 \le t \le s , \\ A(t)\xi & \text{for } t > s . \end{cases}$$

Hence, even though f is not continuous, $|y(t)| \le rM$ for $t \ge 0$. It follows that

$$|X(t)PX^{-1}(s)| \le 1 + rM \quad \text{for } 0 \le s \le t ,$$

$$|X(t)(I - P)X^{-1}(s)| \le rM \quad \text{for } 0 \le t \le s .$$

The result now follows from Lemma 2.

We show finally, by an example, that the hypothesis of bounded growth cannot be omitted in Proposition 3 (nor in Proposition 2.1). Let $\phi(t)$ be a real-valued continuously differentiable function such that $0 < \phi(t) \le 1$ for all $t \ge 0$, $\int_0^\infty \{1/\phi(t) - 1\}dt < \infty$, and $\phi(n)/\phi(n - 2^{-n}) \to \infty$ as $n \to \infty$. Such a function can easily be constructed explicitly. The homogeneous equation

$$x' = [\phi'(t)/\phi(t) - 1]x \tag{7}$$

has the solutions $x(t) = x(0)e^{-t}\phi(t)$. For all $t \ge 0$

$$\begin{aligned} \int_0^t |x(t)/x(s)|ds &= \int_0^t e^{-(t-s)}\phi(t)/\phi(s)ds \\ &\le e^{-t}\phi(t)\int_0^t e^s ds + \int_0^t \{1/\phi(s) - 1\}ds \\ &\le 1 + \int_0^\infty \{1/\phi(s) - 1\}ds . \end{aligned}$$

Hence for any bounded, continuous function $f(t)$ the inhomogeneous equation

$$y' = [\phi'(t)/\phi(t) - 1]y + f(t)$$

has the bounded solution

$$y(t) = \int_0^t \{x(t)/x(s)\}f(s)ds .$$

The equation (7) is asymptotically, and even exponentially, stable, since $\phi(t)$ is bounded. On the other hand it is not uniformly stable, since

$$x(n)/x(n - 2^{-n}) \to \infty \quad \text{as} \quad n \to \infty .$$

It seems desirable to exclude such behaviour.

4. ROUGHNESS

One of the most important properties of exponential dichotomies is their *roughness*. That is, they are not destroyed by small perturbations of the coefficient matrix. This was first proved by Massera and Schäffer (1958) under the assumption that the original coefficient matrix is bounded. Later Schäffer (1963) removed this assumption. Both these proofs depend on the functional analytic characterisation of exponential dichotomies and hence, ultimately, on the closed graph theorem. Coppel (1967) gave a completely elementary proof. The key idea was to show that the general case can be reduced to the much simpler special case in which the coefficient matrix $A(t)$ commutes for every t with the projection P of the exponential dichotomy. This is a useful general principle, but the resulting proof of roughness is somewhat indirect. A direct and elementary proof was given by Daleckiĭ and Kreĭn (1970), but under the assumption that $A(t)$ is bounded. We will show here that this assumption can be quite easily removed.

To avoid interrupting the argument we first prove

LEMMA 1. *Let* $\phi(t)$ *be a bounded, continuous real-valued function such that*

$$\phi(t) \le Ke^{-\alpha t} + \theta\alpha\int_0^\infty e^{-\alpha|t-u|}\phi(u)du \quad \textit{for all} \quad t \ge 0 ,$$

where K , α , θ *are positive constants. If* $\theta < \frac{1}{2}$ *then*

$$\phi(t) \le \rho K e^{-\beta t} \quad \textit{for all} \quad t \ge 0 ,$$

where

$$\beta = \alpha(1 - 2\theta)^{\frac{1}{2}} , \; \rho = \theta^{-1}\{1 - (1 - 2\theta)^{\frac{1}{2}}\} .$$

Proof. Consider the corresponding integral equation

$$\psi(t) = Ke^{-\alpha t} + \theta\alpha\int_0^\infty e^{-\alpha|t-u|}\psi(u)du .$$

By separating the interval of integration $[0 , \infty)$ into two parts, $[0 , t]$

and $[t, \infty)$, we see that any bounded, continuous solution $\psi(t)$ is differentiable and

$$\psi'(t) = -\alpha K e^{-\alpha t} - \theta\alpha^2\int_0^t e^{-\alpha(t-u)}\psi(u)du$$

$$+ \theta\alpha^2\int_t^\infty e^{-\alpha(u-t)}\psi(u)du .$$

Hence $\psi(t)$ is differentiable again and

$$\psi''(t) = \alpha^2 K e^{-\alpha t} - 2\theta\alpha^2\psi(t) + \theta\alpha^3\int_0^\infty e^{-\alpha|t-u|}\psi(u)du .$$

It follows that $\psi(t)$ is a solution of the differential equation

$$\psi'' = \alpha^2(1 - 2\theta)\psi .$$

Since we are assuming that $\psi(t)$ is bounded this implies, if $\theta < \frac{1}{2}$, that

$$\psi(t) = ce^{-\beta t} ,$$

for some constant c. Substituting this expression for $\psi(t)$ back in the integral equation we obtain $c = \rho K$. Thus the integral equation has the unique bounded, continuous solution $\psi(t) = \rho K e^{-\beta t}$.

It is easily verified that, for any constant $L \geq \theta^{-1}K$,

$$L \geq Ke^{-\alpha t} + \theta\alpha\int_0^\infty e^{-\alpha|t-u|}L du \quad \text{for all} \quad t \geq 0 .$$

If we choose $L \geq \sup \phi(t)$ it follows by the method of successive approximations that the integral equation has a solution $\psi(t)$ such that $\phi(t) \leq \psi(t) \leq L$ for all $t \geq 0$. Since $\psi(t)$ is uniquely determined, the result follows.

By applying the preceding lemma to $\phi(s - t)$ we obtain at once

LEMMA 2. *Let* $\phi(t)$ *be a continuous real-valued function such that*

$$\phi(t) \leq Ke^{-\alpha(s-t)} + \theta\alpha\int_0^s e^{-\alpha|t-u|}\phi(u)du \quad \textit{for} \quad 0 \leq t \leq s ,$$

where K, α, θ *are positive constants. If* $\theta < \frac{1}{2}$ *then*

$$\phi(t) \leq \rho K e^{-\beta(s-t)} \quad \textit{for} \quad 0 \leq t \leq s ,$$

where

$$\beta = \alpha(1 - 2\theta)^{\frac{1}{2}} , \quad \rho = \theta^{-1}\{1 - (1 - 2\theta)^{\frac{1}{2}}\} .$$

Now let $A(t)$ be a continuous matrix function for $t \geq 0$ and let $X(t)$ be the fundamental matrix for the linear differential equation

$$x' = A(t)x \tag{1}$$

such that $X(0) = I$. We assume that the equation (1) has an exponential dichotomy. Thus there exists a projection P and positive constants α , K such that

$$
\begin{aligned}
|X(t)PX^{-1}(s)| &\le Ke^{-\alpha(t-s)} \quad \text{for} \quad t \ge s \ , \\
|X(t)(I - P)X^{-1}(s)| &\le Ke^{-\alpha(s-t)} \quad \text{for} \quad s \ge t \ .
\end{aligned} \tag{2}
$$

Let $B(t)$ be a bounded, continuous matrix function. We wish to show that if

$$\delta = \sup_{t\ge 0} |B(t)|$$

is sufficiently small then the perturbed equation

$$y' = [A(t) + B(t)]y \tag{3}$$

also possesses an exponential dichotomy.

For any bounded, continuous matrix function $Y(t)$ set

$$\|Y\| = \sup_{t\ge 0} |Y(t)|$$

and

$$
\begin{aligned}
TY(t) = X(t)P &+ \int_0^t X(t)PX^{-1}(u)B(u)Y(u)du \\
&- \int_t^\infty X(t)(I - P)X^{-1}(u)B(u)Y(u)du \ .
\end{aligned}
$$

Then

$$
\begin{aligned}
|TY(t)| \le Ke^{-\alpha t} &+ K\|Y\|\{\int_0^t e^{-\alpha(t-u)}|B(u)|du \\
&+ \int_t^\infty e^{-\alpha(u-t)}|B(u)|du\} \ .
\end{aligned}
$$

It follows that $TY(t)$ is again bounded and continuous, and

$$\|TY\| \le K + 2\alpha^{-1}K\delta\|Y\| \ .$$

Similarly we obtain

$$\|TY - T\tilde{Y}\| \le 2\alpha^{-1}K\delta\|\tilde{Y} - Y\| \ .$$

Hence, by the contraction principle, if

$$\theta = \alpha^{-1}K\delta < \tfrac{1}{2} \ ,$$

then the mapping T has a unique fixed point. Denoting this fixed point by $Y_1(t)$ we have

$$
\begin{aligned}
Y_1(t) = X(t)P &+ \int_0^t X(t)PX^{-1}(u)B(u)Y_1(u)du \\
&- \int_t^\infty X(t)(I - P)X^{-1}(u)B(u)Y_1(u)du \ .
\end{aligned} \tag{4}
$$

It follows that $Y_1(t)$ is differentiable and is a matrix solution of the differential equation (3). Since $Y_1(t)P$ is also a fixed point of T we have $Y_1(t)P = Y_1(t)$. In particular, if $Q = Y_1(0)$ then

$$QP = Q .$$

From (4), with t replaced by s, we obtain also

$$X(t)PX^{-1}(s)Y_1(s) = X(t)P + \int_0^s X(t)PX^{-1}(u)B(u)Y_1(u)du . \tag{5}$$

Combining this with (4) we obtain

$$\begin{aligned} Y_1(t) = X(t)PX^{-1}(s)Y_1(s) &+ \int_s^t X(t)PX^{-1}(u)B(u)Y_1(u)du \\ &- \int_t^\infty X(t)(I - P)X^{-1}(u)B(u)Y_1(u)du . \end{aligned} \tag{6}$$

On the other hand, setting $t = s = 0$ in (5) we get

$$PQ = P .$$

It follows that $Y_1(t)Q$ is also a fixed point of T and hence

$$Y_1(t)Q = Y_1(t) .$$

For $t = 0$ this shows that Q is a projection.

If $Y(t)$ is the fundamental matrix of the equation (3) such that $Y(0) = I$ then

$$Y_1(t) = Y(t)Q .$$

Put

$$Y_2(t) = Y(t)(I - Q) ,$$

so that $Y(t) = Y_1(t) + Y_2(t)$. By the variation of constants formula,

$$Y_2(t) = X(t)(I - Q) + \int_0^t X(t)X^{-1}(u)B(u)Y_2(u)du . \tag{7}$$

Replacing t by s and using the fact that $(I - P)(I - Q) = I - Q$ we get

$$\begin{aligned} X(t)(I - P)X^{-1}(s)Y_2(s) &= X(t)(I - Q) \\ &+ \int_0^s X(t)(I - P)X^{-1}(u)B(u)Y_2(u)du . \end{aligned}$$

Combining this with the previous equation we obtain

$$Y_2(t) = X(t)(I - P)X^{-1}(s)Y_2(s) + \int_0^t X(t)PX^{-1}(u)B(u)Y_2(u)du$$
$$- \int_t^s X(t)(I - P)X^{-1}(u)B(u)Y_2(u)du \ . \tag{8}$$

It follows from (6) and (8) that, for any vector ξ ,

$$|Y_1(t)\xi| \le Ke^{-\alpha(t-s)}|Y_1(s)\xi| + \theta\alpha\int_s^\infty e^{-\alpha|t-u|}|Y_1(u)\xi|du$$
$$\text{for} \quad t \ge s \ge 0 \ ,$$
$$|Y_2(t)\xi| \le Ke^{-\alpha(s-t)}|Y_2(s)\xi| + \theta\alpha\int_0^s e^{-\alpha|t-u|}|Y_2(u)\xi|du$$
$$\text{for} \quad s \ge t \ge 0 \ .$$

Hence, by Lemmas 1 and 2,

$$|Y_1(t)\xi| \le \rho Ke^{-\beta(t-s)}|Y_1(s)\xi| \quad \text{for} \quad t \ge s \ge 0 \ ,$$
$$|Y_2(t)\xi| \le \rho Ke^{-\beta(s-t)}|Y_2(s)\xi| \quad \text{for} \quad s \ge t \ge 0 \ , \tag{9}$$

where

$$\beta = \alpha(1 - 2\theta)^{\frac{1}{2}} \ , \ \rho = \theta^{-1}\{1 - (1 - 2\theta)^{\frac{1}{2}}\} \ .$$

To deduce from the inequalities (9) that the differential equation (3) has an exponential dichotomy it is only necessary to show that $Y(t)QY^{-1}(t)$ is bounded.

From (4) we immediately obtain

$$X(t)(I - P)X^{-1}(t)Y_1(t) = -\int_t^\infty X(t)(I - P)X^{-1}(u)B(u)Y_1(u)du \ .$$

Since, by (9),

$$|Y_1(u)\xi| \le \rho Ke^{-\beta(u-t)}|Y_1(t)\xi| \quad \text{for} \quad u \ge t$$

it follows that

$$|X(t)(I - P)X^{-1}(t)Y_1(t)\xi| \le \theta\alpha\rho K|Y_1(t)\xi|\int_t^\infty e^{-(\alpha+\beta)(u-t)}du$$
$$= \eta|Y_1(t)\xi| \ , \tag{10}$$

where $\eta = (\rho - 1)K$. From (7) we obtain similarly

$$X(t)PX^{-1}(t)Y_2(t) = \int_0^t X(t)PX^{-1}(u)B(u)Y_2(u)du \ .$$

Since, by (9),

$$|Y_2(u)\xi| \le \rho Ke^{-\beta(t-u)}|Y_2(t)\xi| \quad \text{for} \quad 0 \le u \le t$$

it follows that

$$|X(t)PX^{-1}(t)Y_2(t)\xi| \le \theta\alpha\rho K|Y_2(t)\xi|\int_0^t e^{-(\alpha+\beta)(t-u)}du$$

$$\le \eta|Y_2(t)\xi| \ . \tag{11}$$

Now it is easily verified that

$$Y(t)QY^{-1}(t) - X(t)PX^{-1}(t) = X(t)(I - P)X^{-1}(t)Y(t)QY^{-1}(t)$$

$$- X(t)PX^{-1}(t)Y(t)(I - Q)Y^{-1}(t) \ .$$

Replacing ξ by $Y^{-1}(t)\xi$ in (10) and (11), for an arbitrary vector ξ , we obtain

$$|Y(t)QY^{-1}(t) - X(t)PX^{-1}(t)| \le \eta(\gamma_1 + \gamma_2) \ ,$$

where

$$\gamma_1(t) = |Y(t)QY^{-1}(t)| \ , \ \gamma_2(t) = |Y(t)(I - Q)Y^{-1}(t)| \ .$$

Hence

$$\gamma_1 \le \eta(\gamma_1 + \gamma_2) + K \ .$$

Since

$$Y(t)QY^{-1}(t) - X(t)PX^{-1}(t) = X(t)(I - P)X^{-1}(t)$$

$$- Y(t)(I - Q)Y^{-1}(t) \ ,$$

we have equally

$$\gamma_2 \le \eta(\gamma_1 + \gamma_2) + K \ .$$

Adding, we obtain

$$\gamma_1 + \gamma_2 \le 2(1 - 2\eta)^{-1}K$$

if $\eta < \frac{1}{2}$, and hence

$$\gamma_1 \ , \gamma_2 \le (1 - 2\eta)^{-1}K \ .$$

If in (9) we replace ξ by $Y^{-1}(s)\xi$ we now obtain the exponential dichotomy:

$$|Y(t)QY^{-1}(s)| \le Le^{-\beta(t-s)} \quad \text{for} \quad t \ge s \ge 0 \ ,$$

$$|Y(t)(I - Q)Y^{-1}(s)| \le Le^{-\beta(s-t)} \quad \text{for} \quad s \ge t \ge 0 \ ,$$

where $L = (1 - 2\eta)^{-1}\rho K^2$.

There is no special merit in the values of the constants. The condition $\eta < \frac{1}{2}$ is certainly satisfied if $\theta < 1/4K$. Since necessarily $K \geq 1$, some elementary calculations yield the following simpler final statement:

PROPOSITION 1. *Suppose the linear differential equation* (1) *has an exponential dichotomy* (2) *on* $\mathbb{R}_+$. *If*

$$\delta = \sup_{t\in\mathbb{R}_+} |B(t)| < \alpha/4K^2$$

then the perturbed equation (3) *also has an exponential dichotomy:*

$$|Y(t)QY^{-1}(s)| \leq (5/2)K^2 e^{-(\alpha-2K\delta)(t-s)} \quad \textit{for} \quad t \geq s \geq 0 ,$$

$$|Y(t)(I - Q)Y^{-1}(s)| \leq (5/2)K^2 e^{-(\alpha-2K\delta)(s-t)} \quad \textit{for} \quad s \geq t \geq 0 ,$$

where $Y(t)$ *is the fundamental matrix for* (3) *such that* $Y(0) = I$ *and the projection* Q *has the same nullspace as the projection* P .

Moreover

$$|Y(t)QY^{-1}(t) - X(t)PX^{-1}(t)| \leq 4\alpha^{-1}K^3\delta \quad \textit{for all} \quad t \geq 0 .$$

The roughness of exponential dichotomies on $\mathbb{R}$ can be deduced from this result. Suppose (2) holds on $\mathbb{R}$ and

$$\delta = \sup_{t\in\mathbb{R}} |B(t)| < \alpha/4K^2 .$$

Then the perturbed equation (3) has an exponential dichotomy on each of the half-lines $\mathbb{R}_+$, $\mathbb{R}_-$ with corresponding projections Q' , Q'' . Moreover Q' has the same nullspace as P , $I - Q''$ has the same nullspace as $I - P$, and (by the proof of Proposition 1)

$$|Q' - P| \leq \theta(1 - 2\theta)^{-1}K , \quad |Q'' - P| \leq \theta(1 - 2\theta)^{-1}K ,$$

where $\theta = \alpha^{-1}K\delta < 1/4K$. Thus

$$Q'P = Q' , \quad PQ' = P ,$$

$$Q''P = P , \quad PQ'' = Q'' ,$$

and $|Q' - Q''| < 1$. Put

$$S = I + Q' - Q'' , \quad Q = SPS^{-1} .$$

Then Q is a projection and

$$SP = Q' , \quad S(I - P) = I - Q'' .$$

It follows that Q has the same range as Q' and the same nullspace as Q''. Consequently the equation (3) has an exponential dichotomy on each of the half-lines $\mathbb{R}_+$, $\mathbb{R}_-$ with common projection Q, and therefore an exponential dichotomy on $\mathbb{R}$ with projection Q (and exponent $\alpha - 2K\delta$).

We consider now the roughness of ordinary dichotomies and, to illustrate a different method, we will use their functional-analytic characterisation (Proposition 3.1). We propose to prove

PROPOSITION 2. *Suppose the linear differential equation* (1) *has an ordinary dichotomy on the half-line* $\mathbb{R}_+$. *If*

$$\int_0^\infty |B(t)|dt < \infty$$

then the perturbed equation (3) *also has an ordinary dichotomy on* $\mathbb{R}_+$. *Moreover there exists an invertible matrix* T *such that* (1) *has a dichotomy with projection* P *if and only if* (3) *has a dichotomy with projection* TPT^{-1}.

In fact this will follow from

PROPOSITION 3. *Suppose the linear differential equation* (1) *has an ordinary dichotomy on the half-line* $\mathbb{R}_+$. *If*

$$\int_0^\infty |B(t)|dt < \infty \ , \ \int_0^\infty |f(t)|dt < \infty$$

then there is a one-to-one affine mapping between the bounded solutions of the equation (1) *and the bounded solutions of the inhomogeneous perturbed equation*

$$y' = [A(t) + B(t)]y + f(t) \ , \tag{12}$$

such that the difference between corresponding solutions tends to zero as $t \to \infty$.

Proof. Again let $X(t)$ be the fundamental matrix of (1) such that $X(0) = I$. Then there exists a projection P and a constant $K > 0$ such that

$$|X(t)PX^{-1}(s)| \leq K \quad \text{for} \quad t \geq s \geq 0 \ ,$$

$$|X(t)(I - P)X^{-1}(s)| \leq K \quad \text{for} \quad s \geq t \geq 0 \ .$$

Moreover, by Proposition 2.2, we can suppose that $|X(t)P\xi| \to 0$ as $t \to \infty$ for any vector ξ.

Choose $t_0 \geq 0$ so large that

$$\theta = K\int_{t_0}^\infty |B(t)|dt < 1 \ .$$

For any continuous function $y(t)$ with

$$\|y\| = \sup_{t \geq t_0} |y(t)| < \infty$$

set

$$Ty(t) = \int_{t_0}^{t} X(t)PX^{-1}(s)[B(s)y(s) + f(s)]ds$$

$$- \int_{t}^{\infty} X(t)(I - P)X^{-1}(s)[B(s)y(s) + f(s)]ds \ .$$

Then $Ty(t)$ is again bounded and continuous. Moreover, for any two bounded continuous functions $y_1(t)$, $y_2(t)$ we have

$$\|Ty_1 - Ty_2\| \leq K\int_{t_0}^{\infty} |B(s)||y_1(s) - y_2(s)|ds$$

$$\leq \theta\|y_1 - y_2\| \ .$$

Let $x(t)$ be any bounded solution of the differential equation (1). Then by the contraction principle, the integral equation

$$y(t) = x(t) + Ty(t) \tag{13}$$

has a unique bounded continuous solution $y(t)$. It is easily verified that $y(t)$ is a solution of the differential equation (12). Conversely, if $y(t)$ is a bounded solution of (12) then $x(t) = y(t) - Ty(t)$ is a bounded solution of (1).

Thus (13) establishes a 1-1 affine mapping between the bounded solutions of (12) and (1). Given any $\varepsilon > 0$ we can choose $t_1 \geq t_0$ so that

$$K\int_{t_1}^{\infty} [|B(s)||y(s)| + |f(s)|]ds < \varepsilon \ .$$

Then

$$|Ty(t)| \leq |X(t)P\int_{t_0}^{t_1} X^{-1}(s)[B(s)y(s) + f(s)]ds| + \varepsilon$$

$$< 2\varepsilon \quad \text{for all large } t \ .$$

Thus $y(t) - x(t) \to 0$ as $t \to \infty$.

Proposition 3 shows incidentally, for $B(t) \equiv 0$, that if the homogeneous equation (1) has an ordinary dichotomy on $\mathbb{R}_+$ then for any function $f(t)$ such that

$$\int_0^{\infty} |f(t)|dt < \infty$$

the inhomogeneous equation

$$y' = A(t)y + f(t)$$

has at least one bounded solution *which tends to zero* as $t \to \infty$

5. DICHOTOMIES AND REDUCIBILITY

Consider the linear differential equation

$$x' = A(t)x \, , \tag{1}$$

where the coefficient matrix $A(t)$ is continuous on an interval J. The equation (1) is said to be *kinematically similar* to another equation

$$y' = B(t)y \tag{2}$$

if there exists a continuously differentiable invertible matrix $S(t)$ which satisfies the differential equation

$$S' = A(t)S - SB(t)$$

and which is bounded, together with its inverse, on J. The change of variables $x = S(t)y$ then transforms (1) into (2).

We will say that the equation (1) is *reducible* if it is kinematically similar to an equation (2) whose coefficient matrix $B(t)$ has the block form

$$\begin{pmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{pmatrix} ,$$

$B_1(t)$ and $B_2(t)$ being matrices of lower order than $B(t)$. In coordinate-free terms this means that there is a projection $P \neq 0$, I which commutes with $B(t)$ for every $t \in J$. It may be noted that although this definition agrees with the definition of reducibility in linear algebra, it differs from Lyapunov's use of the term. (Lyapunov calls an equation (1) reducible if it is kinematically similar to an autonomous equation.)

We are going to show that if the equation (1) has an ordinary or exponential dichotomy with projection $P \neq 0$, I then it is reducible. We will denote by $\|\xi\|$ the *Euclidean* norm of the vector ξ. Then the induced matrix norm $\|A\|$ is the square root of the largest eigenvalue of A^*A. A projection P is said to be *orthogonal* if $P^* = P$.

LEMMA 1. *Let* P *be an orthogonal projection and let* X *be an invertible matrix. Then there exists an invertible matrix* S *such that*

$$SPS^{-1} = XPX^{-1}$$

and

$$\|S\| \le 2^{\frac{1}{2}} ,$$

$$\|S^{-1}\| \le \{\|XPX^{-1}\|^2 + \|X(I - P)X^{-1}\|^2\}^{\frac{1}{2}} .$$

Moreover, if $X = X(t)$ *is a continuous, or continuously differentiable, function of* t *on an interval* J *we can take* $S = S(t)$ *to be a continuous, or continuously differentiable, function of* t *on the same interval.*

Proof. Since any positive Hermitian matrix has a unique positive square root, there exists a unique $R = R^* > 0$ such that

$$R^2 = PX^*XP + (I - P)X^*X(I - P) .$$

Moreover, since R^2 commutes with P , so also does R . Thus if we put $S = XR^{-1}$ then S has the first required property. Since

$$I = PS^*SP + (I - P)S^*S(I - P)$$

we have, for any vector ξ ,

$$\begin{aligned}\|S\xi\|^2 &\le \{\|SP\xi\| + \|S(I - P)\xi\|\}^2 \\ &\le 2\{\|SP\xi\|^2 + \|S(I - P)\xi\|^2\} \\ &= 2\|\xi\|^2 .\end{aligned}$$

Thus $\|S\| \le 2^{\frac{1}{2}}$. On the other hand,

$$(S^{-1})^*S^{-1} = X^{*-1}PX^*XPX^{-1} + X^{*-1}(I - P)X^*X(I - P)X^{-1}$$

and hence

$$\|S^{-1}\|^2 \le \|XPX^{-1}\|^2 + \|X(I - P)X^{-1}\|^2 .$$

The last remark follows from the fact that the positive square root of a continuous, or continuously differentiable, positive Hermitian matrix function is again continuous, or continuously differentiable.

LEMMA 2. *Let* $X(t)$ *be a fundamental matrix for the linear differential equation* (1), *and suppose there exists an orthogonal projection* P *such that* $X(t)PX^{-1}(t)$ *is bounded for* $t \in J$.

Then the equation (1) *is kinematically similar to an equation*

$$z' = C(t)z \tag{3}$$

whose coefficient matrix $C(t)$ *is Hermitian, commutes with* P , *and satisfies*

$$\|C(t)\| \le \|A(t)\| \quad \textit{for every} \quad t \in J \ .$$

Proof. Under the present circumstances the functions $R(t)$ and $S(t)$ of the previous lemma are continuously differentiable. The change of variables $x = S(t)y$ transforms (1) into (2) , where $B = S^{-1}(AS - S')$. Since (2) has $R(t)$ as a fundamental matrix, $B = R'R^{-1}$ commutes with P .

Let $U(t)$ be the fundamental matrix for the equation

$$u' = \tfrac{1}{2}[B(t) - B^*(t)]u \tag{4}$$

such that $U(t_0) = I$ for some $t_0 \in J$. Then $U(t)$ is unitary for every $t \in J$, since the coefficient matrix is skew-Hermitian. Moreover $U(t)$ commutes with P for every $t \in J$, since $B^*(t)$ commutes with $P^* = P$ and the solutions of (4) are uniquely determined by their initial values. It is easily verified that the further change of variables $y = U(t)z$ transforms (2) into (3), where

$$C(t) = \tfrac{1}{2}U^{-1}(t)[B(t) + B^*(t)]U(t)$$

is Hermitian and commutes with P for every $t \in J$.

For each fixed t there exist real numbers λ , μ such that

$$\lambda I \le A + A^* \le \mu I \ .$$

If λ has its greatest value and μ its least value then

$$\|A + A^*\| = \max\{|\lambda| \ , \ |\mu|\} \ .$$

From the definition of R we have

$$RR' + R'R = PX^*(A + A^*)XP$$
$$+ (I - P)X^*(A + A^*)X(I - P) \ .$$

It follows that

$$\lambda R^2 \le RR' + R'R \le \mu R^2$$

and hence

$$\lambda I \le R'R^{-1} + R^{-1}R' \le \mu I \ .$$

Therefore

$$\|B + B^*\| \le \|A + A^*\| .$$

Since U is unitary and $\|A^*\| = \|A\|$ this implies

$$\|C\| = \tfrac{1}{2}\|B + B^*\| \le \|A\| .$$

Finally, we have $x = T(t)z$, where $T(t) = S(t)U(t)$ satisfies

$$\|T(t)\| \le 2^{\frac{1}{2}} ,$$

$$\|T^{-1}(t)\| \le \{\|X(t)PX^{-1}(t)\|^2 + \|X(t)(I - P)X^{-1}(t)\|^2\}^{\frac{1}{2}} .$$

It follows at once from the definitions that kinematic similarity preserves an exponential or ordinary dichotomy. Lemma 2 provides the following more precise result:

LEMMA 3. *If the linear differential equation* (1) *has an exponential or ordinary dichotomy*

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\le Ke^{-\alpha(t-s)} \quad \textit{for} \quad t \ge s , \\ \|X(t)(I - P)X^{-1}(s)\| &\le Ke^{-\alpha(s-t)} \quad \textit{for} \quad s \ge t , \end{aligned} \tag{5}$$

where $P = P^*$, *then there exists a continuously differentiable invertible matrix* $T(t)$ *with*

$$\|T(t)\| \le 2^{\frac{1}{2}} , \ \|T^{-1}(t)\| \le 2^{\frac{1}{2}}K ,$$

such that the change of variables $x = T(t)z$ *transforms* (1) *into an equation* (3) *whose coefficient matrix* $C(t)$ *is Hermitian, commutes with* P *and satisfies* $\|C(t)\| \le \|A(t)\|$ *for every* $t \in J$.

The fundamental matrix $Z(t) = T^{-1}(t)X(t)$ *of the equation* (3) *commutes with* P *for every* $t \in J$ *and satisfies*

$$\|Z(t)PZ^{-1}(s)\| \le 2K^2e^{-\alpha(t-s)} \quad \textit{for} \quad t \ge s ,$$

$$\|Z(t)(I - P)Z^{-1}(s)\| \le 2K^2e^{-\alpha(s-t)} \quad \textit{for} \quad s \ge t .$$

The advantage of this transformation is that it eliminates the interaction of the 'big' solutions with the 'small' solutions. It will now be used to establish the roughness of exponential dichotomies on an arbitrary interval. It is sufficient to consider the case where the projection P is orthogonal, since any projection is similar to an orthogonal projection and, in fact, to a uniquely determined matrix of the form

$$\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix},$$

where $0 \le k \le n$.

PROPOSITION 1. *Suppose the linear differential equation* (1) *has an exponential dichotomy* (5) *, where* $P = P^*$ *and* $K \ge 1$, $\alpha > 0$.

If

$$\delta = \sup_{t \in J} \|B(t)\| \le \alpha/36K^5$$

then the perturbed equation

$$y' = [A(t) + B(t)]y \tag{6}$$

also has an exponential dichotomy:

$$\|Y(t)PY^{-1}(s)\| \le 12K^3 e^{-(\alpha - 6K^3\delta)(t-s)} \quad \textit{for} \quad t \ge s ,$$

$$\|Y(t)(I - P)Y^{-1}(s)\| \le 12K^3 e^{-(\alpha - 6K^3\delta)(s-t)} \quad \textit{for} \quad s \ge t .$$

Moreover, for all $t \in J$,

$$\|Y(t)PY^{-1}(t) - X(t)PX^{-1}(t)\| \le 144\alpha^{-1}K^6\delta .$$

Proof. We make a linear change of variables $x = T(t)z$, as in Lemma 3. This transforms (1) into (3) and the perturbed equation (6) into

$$w' = [C(t) + D(t)]w , \tag{7}$$

where $D(t) = T^{-1}(t)B(t)T(t)$. Thus $\|D(t)\| \le 2K\delta$. Our object now is to transform the equation (7) into a kinematically similar equation whose coefficient matrix commutes with P .

For any matrix E put

$$E_1 = PEP + (I - P)E(I - P) ,$$

$$E_2 = PE(I - P) + (I - P)EP ,$$

so that $E = E_1 + E_2$. Evidently E_1 commutes with P . We look for a change of variables $w = S(t)v$ which transforms (7) into

$$v' = [C(t) + \{D(t)S(t)\}_1]v . \tag{8}$$

This will be achieved if $S(t)$ satisfies the differential equation

$$S' = C(t)S - SC(t) + D(t)S - S\{D(t)S\}_1 \ , \tag{9}$$

or putting $S(t) = I + H(t)$, if $H(t)$ satisfies the equation

$$H' = C(t)H - HC(t) + \{D(t)(I + H)\}_2 - H\{D(t)(I + H)\}_1 \ . \tag{10}$$

Consider the integral equation $H = TH$, where

$$TH(t) = \int_a^t Z(t)PZ^{-1}(s)[I - H(s)]D(s)[I + H(s)]Z(s)(I - P)Z^{-1}(t)ds$$

$$- \int_t^b Z(t)(I - P)Z^{-1}(s)[I - H(s)]D(s)[I + H(s)]Z(s)PZ^{-1}(t)ds$$

and $J = (a , b)$. For any bounded continuous matrix function $H(t)$ set

$$\|H\| = \sup_{t \in J} \|H(t)\| \ .$$

Using Lemma 3 and the identity

$$(I - G)D(I + G) - (I - H)D(I + H)$$

$$= (H - G)D - D(H - G) + (H - G)DH + GD(H - G)$$

we see that if $\|H\| \le \frac{1}{2}$, $\|G\| \le \frac{1}{2}$ then

$$\|TH(t) - TG(t)\| \le 12K^4\|H - G\|\{\int_a^t e^{-2\alpha(t-s)}\|D(s)\|ds$$

$$+ \int_t^b e^{-2\alpha(s-t)}\|D(s)\|ds\}$$

and hence

$$\|TH - TG\| \le 12\alpha^{-1}K^4\|H - G\|\|D\| \ .$$

Similarly for $\|H\| \le \frac{1}{2}$ we obtain

$$\|TH(t)\| \le 9K^4\{\int_a^t e^{-2\alpha(t-s)}\|D(s)\|ds + \int_t^b e^{-2\alpha(s-t)}\|D(s)\|ds\}$$

and hence

$$\|TH\| \le 9\alpha^{-1}K^4\|D\| \ .$$

Thus if $\delta \le \alpha/36K^5$ then T is a contraction and maps the ball $\|H\| \le \frac{1}{2}$ into itself. Consequently, by the contraction principle, the integral equation $H = TH$ has a unique solution $H(t)$ in $\|H\| \le \frac{1}{2}$.

Moreover

$$\|H\| \le 18\alpha^{-1}K^5\delta \ .$$

The solution $H(t)$ is differentiable and satisfies the differential equation

$$H' = C(t)H - HC(t) + \{(I - H)D(t)(I + H)\}_2 .$$

Moreover $H(t) = H$ satisfies

$$PHP = 0 , (I - P)H(I - P) = 0 .$$

Hence $H = PH + HP$, from which it follows that $H(t)$ is also a solution of the differential equation (10). Then $S(t) = I + H(t)$ satisfies (9) and $\|S\| \le 3/2$, $\|S^{-1}\| \le 2$.

Since the coefficient matrices of (3) and (8) commute with P these equations decompose into two independent systems to which we can apply Proposition 1.1, or Proposition 1.1 with t replaced by $-t$. Since

$$\|\{D(t)S(t)\}_1\| \le \|D(t)S(t)\| \le 3K\delta$$

it follows that the equation (8) has a fundamental matrix $V(t)$ satisfying

$$\|V(t)PV^{-1}(s)\| \le 2K^2 e^{-(\alpha-6K^3\delta)(t-s)} \quad \text{for} \quad t \ge s ,$$

$$\|V(t)(I - P)V^{-1}(s)\| \le 2K^2 e^{-(\alpha-6K^3\delta)(s-t)} \quad \text{for} \quad s \ge t .$$

Hence $Y(t) = T(t)S(t)V(t)$ is a fundamental matrix for (6) satisfying

$$\|Y(t)PY^{-1}(s)\| \le 12K^3 e^{-(\alpha-6K^3\delta)(t-s)} \quad \text{for} \quad t \ge s ,$$

$$\|Y(t)(I - P)Y^{-1}(s)\| \le 12K^3 e^{-(\alpha-6K^3\delta)(s-t)} \quad \text{for} \quad s \ge t .$$

Moreover

$$Y(t)PY^{-1}(t) - X(t)PX^{-1}(t) = T(t)[S(t)PS^{-1}(t) - P]T^{-1}(t)$$

and, since $\|H\| \le \frac{1}{2}$,

$$\|S(t)PS^{-1}(t) - P\| \le 4\|H(t)\| .$$

It follows that

$$\|Y(t)PY^{-1}(t) - X(t)PX^{-1}(t)\| \le 144\alpha^{-1}K^6\delta .$$

The constants in Proposition 1 are cruder than those in Proposition 4.1 and, as it stands, there is also a restriction on the norm and the projection. However, Proposition 1 applies to exponential dichotomies on *arbitrary* intervals.

The proof shows too — cf. Lemma 3.1 — that if (1) has an exponential dichotomy on J and if, for some fixed $h > 0$,

$$\bar{\delta} = \sup_{t \in J} h^{-1}\int_t^{t+h} |B(s)|ds$$

is sufficiently small, then (6) has an exponential dichotomy on J with the same projection. Finally, by appealing to Proposition 1.6 instead of to Proposition 1.1, we can prove

PROPOSITION 2. *Let* $A(t)$ *and* $B(t)$ *be continuous matrix functions on an interval* J *such that*

$$|A(t)| \le M \ , \ |B(t)| \le M \ ,$$

and suppose there exist constants $K \ge 1$, $\alpha > 0$ *and a projection* P *such that*

$$|X(t)PX^{-1}(s)| \le Ke^{-\alpha(t-s)} \quad \text{for} \quad t \ge s \ ,$$

$$|X(t)(I - P)X^{-1}(s)| \le Ke^{-\alpha(s-t)} \quad \text{for} \quad s \ge t \ ,$$

where $X(t)$ *is a fundamental matrix for the linear equation* (1). *Then for any positive constant* $\varepsilon < \alpha$ *there exist positive constants* $T = T(P , K)$ *and* $\delta = \delta(P, K, M, \alpha, \varepsilon)$ *such that if*

$$\left|\int_{t_1}^{t_2} B(t)dt\right| \le \delta \quad \text{for} \quad |t_2 - t_1| \le \varepsilon^{-1}T$$

then

$$|Y(t)PY^{-1}(s)| \le \tilde{K}e^{-(\alpha-\varepsilon)(t-s)} \quad \text{for} \quad t \ge s \ ,$$

$$|Y(t)(I - P)Y^{-1}(s)| \le \tilde{K}e^{-(\alpha-\varepsilon)(s-t)} \quad \text{for} \quad s \ge t \ ,$$

where $Y(t)$ *is a fundamental matrix for the perturbed equation* (6) *and* $\tilde{K} = \tilde{K}(P , K) \ge 1$.

As an immediate application we obtain

PROPOSITION 3. *Let* $A(t)$ *be an almost periodic matrix function and suppose that all eigenvalues of its mean value*

$$A_0 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A(t)dt$$

have real part different from zero. In fact, let A_0 *have* k *eigenvalues with real part* $< -\alpha < 0$ *and* $n - k$ *eigenvalues with real part* $> \beta > 0$.

Then for all large $\omega > 0$ *the equation*

$$x' = A(\omega t)x$$

has a fundamental matrix $X(t)$ *such that*

$$|X(t)\tilde{P}X^{-1}(s)| \le Ke^{-\alpha(t-s)} \quad \text{for} \quad t \ge s \ ,$$

$$|X(t)(I - \tilde{P})X^{-1}(s)| \le Le^{-\beta(s-t)} \quad \text{for} \quad s \ge t \ ,$$

where K , L *are positive constants independent of* ω *and*

$$\tilde{P} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} .$$

6. CRITERIA FOR AN EXPONENTIAL DICHOTOMY

Proposition 1.5, with $\alpha < 0$, gives a sufficient condition for uniform asymptotic stability. We intend to derive an analogous sufficient condition for an exponential dichotomy. We first establish some preliminary results.

LEMMA 1. *There exists a numerical constant* $k_n > 0$ *such that, for every non-singular* $n \times n$ *matrix* A ,

$$|A^{-1}| \leq k_n |A|^{n-1} \big/ |\det A| .$$

Proof. Let

$$\det(\lambda I - A) = \lambda^n - a_1 \lambda^{n-1} + \dots + (-1)^n a_n$$
$$= (\lambda - \lambda_1) \dots (\lambda - \lambda_n) .$$

Since

$$a_1 = \sum_j \lambda_j \ , \ a_2 = \sum_{j<k} \lambda_j \lambda_k , \ \dots \ , \ a_n = \lambda_1 \dots \lambda_n$$

and $|\lambda_j| \leq |A|$ we have $|a_k| \leq \binom{n}{k} |A|^k$. But, by the Cayley-Hamilton theorem,

$$a_n A^{-1} = (-A)^{n-1} + a_1 (-A)^{n-2} + \dots + a_{n-1} I .$$

Since $a_n = \det A$ it follows that

$$|\det A| |A^{-1}| \leq |A|^{n-1} \sum_{k=0}^{n-1} \binom{n}{k} = (2^n - 1) |A|^{n-1} .$$

Thus we can take $k_n = 2^n - 1$.

LEMMA 2. *There exists a numerical constant* $h_n > 0$ *such that, if* A *is an* $n \times n$ *matrix which has* k *eigenvalues with real part* $\leq -\alpha$ *and* $n - k$ *eigenvalues with real part* $\geq \alpha$ *, for some* $\alpha > 0$ *, then any matrix* B *satisfying*

$$|B - A| \leq h_n \varepsilon (\varepsilon/|A|)^{n-1} ,$$

where $0 < \varepsilon < \alpha$ *, has* k *eigenvalues with real part* $\leq -\alpha + \varepsilon$ *and* $n - k$ *eigenvalues with real part* $\geq \alpha - \varepsilon$.

Proof. If $|R\lambda| < \alpha - \varepsilon$ and $|\lambda| \leq 2|A|$ then, by Lemma 1,

$$|(A - \lambda I)^{-1}| < k_n (3|A|)^{n-1}/\varepsilon^n .$$

If $|\lambda| > 2|A|$ then

$$\begin{aligned} |(A - \lambda I)^{-1}| &= |\lambda|^{-1} |(I - \lambda^{-1} A)^{-1}| \\ &< (2|A|)^{-1} (1 + 2^{-1} + 2^{-2} + \ldots) \\ &= |A|^{-1} \\ &< k_n (3|A|)^{n-1}/\varepsilon^n , \end{aligned}$$

since $\varepsilon < \alpha \leq |A|$ and $k_n \geq 1$. It follows that if

$$|B - A| \leq \varepsilon^n / k_n (3|A|)^{n-1}$$

then $B - \lambda I$ is invertible for $|R\lambda| < \alpha - \varepsilon$, since

$$|(B - \lambda I) - (A - \lambda I)| = |B - A| < |(A - \lambda I)^{-1}|^{-1} .$$

Thus all eigenvalues of B have real part $\leq -\alpha + \varepsilon$ or $\geq \alpha - \varepsilon$. If $B(\theta) = \theta B + (1 - \theta)A$ then

$$|B(\theta) - A| \leq |B - A| \quad \text{for} \quad 0 \leq \theta \leq 1 .$$

Since the eigenvalues of $B(\theta)$ are continuous functions of θ it follows that A and B have the same number of eigenvalues in the left half-plane.

LEMMA 3. *There exists a numerical constant* $c_n > 0$ *such that, if* A *is an* $n \times n$ *matrix whose eigenvalues all have real part* $\leq -\alpha$ or $\geq \alpha$ *, for some* $\alpha > 0$ *, and if* P *is its spectral projection for the left half-plane then*

$$|P| \leq c_n (\alpha^{-1} |A|)^{n-1} .$$

Proof. We assume $n > 1$, since the result is trivial for $n = 1$. We have

$$P = \frac{1}{2\pi i}\int_\gamma (zI - A)^{-1}dz ,$$

where γ is any rectifiable simple closed curve in the left half-plane which contains in its interior all eigenvalues of A with negative real part. We will take $\gamma = \gamma_1 \cup \gamma_2$, where γ_1 is the left half of the circle $|z| = 2|A|$ and γ_2 is a segment of the imaginary axis. On γ_1 we have

$$\begin{aligned} |(zI - A)^{-1}| &= |z|^{-1}|(I - z^{-1}A)^{-1}| \\ &\le 2^{-1}|A|^{-1}(1 + 2^{-1} + 2^{-2} + \ldots) \\ &= |A|^{-1} . \end{aligned}$$

Therefore

$$|\int_{\gamma_1}(zI - A)^{-1}dz| \le \pi \, . \, 2|A| \, . \, |A|^{-1} = 2\pi .$$

On the other hand, by Lemma 1,

$$\begin{aligned} |(zI - A)^{-1}| &\le k_n|zI - A|^{n-1}\Big/|\det(zI - A)| \\ &= k_n|zI - A|^{n-1}\Big/\prod_{j=1}^{n}|z - \lambda_j| , \end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A . Therefore

$$|\int_{\gamma_2}(zI - A)^{-1}dz| \le k_n(3|A|)^{n-1}\int_{-\infty}^{\infty}\prod_{j=1}^{n}|iy - \lambda_j|^{-1}dy .$$

But $|iy - \lambda_j| \ge \alpha$ and, by Schwarz's inequality,

$$\int_{-\infty}^{\infty}\prod_{j=1}^{2}|iy - \lambda_j|^{-1}dy \le \int_{-\infty}^{\infty}(\alpha^2 + y^2)^{-1}dy = \alpha^{-1}\pi .$$

Hence

$$|\int_{\gamma_2}(zI - A)^{-1}dz| \le \pi k_n(3\alpha^{-1}|A|)^{n-1} .$$

Thus for the projection P we obtain finally

$$|P| \le 1 + \tfrac{1}{2}\, 3^{n-1}k_n(\alpha^{-1}|A|)^{n-1} .$$

Since $\alpha \le |A|$, this inequality can be written in the required form.

Suppose now that $A(t)$ is a continuously differentiable matrix function satisfying the conditions of Lemma 3 for every t in some interval J . Then from the formula

$$P'(t) = \frac{1}{2\pi i}\int_\gamma [zI - A(t)]^{-1}A'(t)[zI - A(t)]^{-1}dz$$

we obtain in the same way

$$|P'(t)| \le d_n(\alpha^{-1}|A(t)|)^{2n-1}|A'(t)| \big/ |A(t)| \ ,$$

where $d_n > 0$ is a numerical constant.

We are now ready for our main result.

PROPOSITION 1. *Let* $A(t)$ *be a continuous* $n \times n$ *matrix function defined on an interval* J *such that*

(i) $A(t)$ *has* k *eigenvalues with real part* $\le -\alpha < 0$ *and* $n - k$ *eigenvalues with real part* $\ge \beta > 0$ *for all* $t \in J$,

(ii) $|A(t)| \le M$ *for all* $t \in J$.

For any positive constant $\varepsilon < \min(\alpha , \beta)$ *there exists a positive constant* $\delta = \delta(M , \alpha + \beta , \varepsilon)$ *such that, if*

$$|A(t_2) - A(t_1)| \le \delta \quad \textit{for} \quad |t_2 - t_1| \le h \ ,$$

where $h > 0$ *is a fixed number not greater than the length of* J , *then the equation*

$$x' = A(t)x \tag{1}$$

has a fundamental matrix $X(t)$ *satisfying the inequalities*

$$\begin{aligned} |X(t)\tilde{P}X^{-1}(s)| &\le Ke^{-(\alpha-\varepsilon)(t-s)} \quad \textit{for} \quad t \ge s \ , \\ |X(t)(I - \tilde{P})X^{-1}(s)| &\le Le^{-(\beta-\varepsilon)(s-t)} \quad \textit{for} \quad s \ge t \ , \end{aligned} \tag{2}$$

where K , L *are positive constants depending only on* $M , \alpha + \beta , \varepsilon$ *and*

$$\tilde{P} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} .$$

Proof. The change of variables $\tilde{x} = e^{(\alpha-\beta)t/2}x$ replaces $A(t)$ by $A(t) + \frac{1}{2}(\alpha - \beta)I$, α and β by $\frac{1}{2}(\alpha + \beta)$, and M by $2M$. We may therefore assume from the outset that $\beta = \alpha$.

Suppose first that $A(t)$ is continuously differentiable with $|A'(t)| \le \delta$ for all $t \in J$. By Proposition 1.5 we may assume that $0 < k < n$. Let $P(t)$ be the projection corresponding to $A(t)$ as in Lemma 3. Then $P(t)$ is continuously differentiable and

$$|P(t)| \le c_n(\alpha^{-1}|A(t)|)^{n-1}$$

$$|P'(t)| \le d_n(\alpha^{-1}|A(t)|)^{2n-1}|A'(t)| \Big/ |A(t)| \; .$$

Since $P(t)$ is a projection we have

$$P' = P'P + PP' \; , \; PP'P = 0 \; . \tag{3}$$

Let $W(t)$ be the solution of the linear differential equation

$$W' = [P'(t)P(t) - P(t)P'(t)]W \tag{4}$$

which takes the value I at a fixed point $t_0 \in J$. Then $W(t)$ is invertible for every $t \in J$. Using (3), one readily verifies that $P(t)W(t)$ is also a solution of (4). Therefore, since the solutions of (4) are uniquely determined by their initial values,

$$P(t)W(t) = W(t)P_0 \; ,$$

where $P_0 = P(t_0)$. Thus $P(t) = W(t)P_0W^{-1}(t)$. Since P_0 is similar to $\tilde{P}$ and $0 < k < n$, it follows from Lemmas 5.1 and 5.2 that there exists a continuously differentiable invertible matrix function $T(t)$ such that

$$P(t) = T(t)\tilde{P}T^{-1}(t) \; ,$$

with $|T(t)||T^{-1}(t)| \le a|P(t)|$, where a is a positive constant depending only on the norm (here and in what follows). Moreover, if $B(t)$ is the coefficient matrix of (4), then

$$C(t) = T^{-1}(t)B(t)T(t) - T^{-1}(t)T'(t)$$

satisfies $|C(t)| \le a|B(t)|$. It follows that

$$|T^{-1}(t)T'(t)| \le a|P(t)||P'(t)| \; .$$

If in equation (1) we make the change of variables $x = T(t)y$ we obtain the equation

$$y' = [D(t) - T^{-1}(t)T'(t)]y \; , \tag{5}$$

where $D(t) = T^{-1}(t)A(t)T(t)$. Since $A(t)$ commutes with $P(t)$, $D(t)$ commutes with $\tilde{P}$. Thus the equation

$$z' = D(t)z \tag{6}$$

decomposes into a system of order k and a system of order $n - k$. Moreover $D(t)$ has the same eigenvalues as $A(t)$ and has $\tilde{P}$ as spectral projection for the left half-plane. Consequently we can apply Proposition 1.5, or Proposition 1.5 with t replaced by $-t$, to the two systems into which (6) splits. Since

$$D' = -T^{-1}T'D + T^{-1}A'T + DT^{-1}T' \ ,$$

we have

$$|D'(t)| \le a(\alpha^{-1}|A(t)|)^{4n-3}|A'(t)| \ .$$

It follows that there exists a positive constant $\delta_1 = \delta_1(M, \alpha, \varepsilon)$ such that, if $\delta \le \delta_1$, the equation (6) has a fundamental matrix $Z(t)$ satisfying

$$|Z(t)\tilde{P}Z^{-1}(s)| \le K_1 e^{-(\alpha-\varepsilon/2)(t-s)} \quad \text{for} \quad t \ge s \ ,$$

$$|Z(t)(I - \tilde{P})Z^{-1}(s)| \le K_1 e^{-(\alpha-\varepsilon/2)(s-t)} \quad \text{for} \quad s \ge t \ ,$$

where $K_1 = K_1(M, \alpha, \varepsilon)$. Therefore, by the roughness of exponential dichotomies (Proposition 5.1), there exists a positive constant $\delta_2 = \delta_2(M, \alpha, \varepsilon) \le \delta_1$ such that, if $\delta \le \delta_2$, the equation (5) has a fundamental matrix $Y(t)$ satisfying

$$|Y(t)\tilde{P}Y^{-1}(s)| \le K_2 e^{-(\alpha-\varepsilon)(t-s)} \quad \text{for} \quad t \ge s \ ,$$

$$|Y(t)(I - \tilde{P})Y^{-1}(s)| \le K_2 e^{-(\alpha-\varepsilon)(s-t)} \quad \text{for} \quad s \ge t \ ,$$

where $K_2 = K_2(M, \alpha, \varepsilon)$. Since the original equation (1) is kinematically similar to the equation (5), this proves the result under the present hypotheses.

We consider now the general case. Put

$$A_1(t) = h^{-1}\int_t^{t+h} A(u)du \ , \ A_2(t) = A(t) - A_1(t) \ .$$

Then $A_1(t)$ is continuously differentiable, $|A_1(t)| \le M$, and $|A_1'(t)| \le \delta h^{-1}$ for all $t \in J$. On the other hand, $A_2(t)$ is continuous and $|A_2(t)| \le \delta$ for all $t \in J$.

By Lemma 2 there exists a positive constant $\delta_3 = \delta_3(M, \varepsilon)$ such that, if $\delta \le \delta_3$, then $A_1(t)$ has k eigenvalues with real part $\le -\alpha + \varepsilon/4$ and $n - k$ eigenvalues with real part $\ge \alpha - \varepsilon/4$ for all $t \in J$. Hence, by what we have already proved, if $\delta \le \delta_4(M, \alpha, \varepsilon)$ the equation

$$x' = A_1(t)x$$

has a fundamental matrix $X_1(t)$ satisfying

$$|X_1(t)\tilde{P}X_1^{-1}(s)| \le K'e^{-(\alpha-\varepsilon/2)(t-s)} \quad \text{for} \quad t \ge s ,$$

$$|X_1(t)(I - \tilde{P})X^{-1}(s)| \le K'e^{-(\alpha-\varepsilon/2)(s-t)} \quad \text{for} \quad s \ge t ,$$

where $K' = K'(M , \alpha , \varepsilon)$. Using again the roughness of exponential dichotomies we see that if $\delta = \delta_0(M , \alpha , \varepsilon)$ then the original equation (1) has a fundamental matrix $X(t)$ satisfying the inequalities (2).

Proposition 1 admits a converse. In proving it we will use the following lemma, which depends on a result that will be established (independently) later.

LEMMA 4. *Let* P *be an orthogonal projection of rank* k . *If* T *is an invertible matrix such that*

$$\|PT\| < 2^{-\frac{1}{2}} , \ \|(I - P)T^{-1}\| < 2^{-\frac{1}{2}} ,$$

then T *has* k *eigenvalues of absolute value less than* 1 *and the remaining* $n - k$ *eigenvalues of absolute value greater than* 1 .

Proof. For any non-zero vector ξ ,

$$\xi^*T^*PT\xi < \tfrac{1}{2}\xi^*\xi ,$$

$$\xi^*(I - P)\xi < \tfrac{1}{2}\xi^*T^*T\xi .$$

Put $H = 2P - I$. Then

$$\begin{aligned} \xi^*T^*HT\xi &= 2\xi^*T^*PT\xi - \xi^*T^*T\xi \\ &< \xi^*\xi - 2\xi^*(I - P)\xi \\ &= \xi^*H\xi . \end{aligned}$$

Thus $T^*HT < H$. Therefore, since H has k eigenvalues 1 and $n - k$ eigenvalues -1 , we can complete the proof by appealing to

LEMMA 5. *Let the matrix* T *be such that, for some Hermitian matrix* H ,

$$T^*HT < H .$$

Then T *has no eigenvalues on the unit circle,* H *is non-singular, and the number of positive (negative) eigenvalues of* H *is equal to the number of eigenvalues of* T *with absolute value less than (greater than)* 1 .

Proof. Let λ be a complex number of absolute value 1 which is not an eigenvalue of T . By replacing T by $\lambda^{-1}T$ we may suppose that 1 is not an eigenvalue of T . Then

$$A = (T + I)(T - I)^{-1}$$

is defined and $HA + A^*H < 0$. Since the linear fractional transformation $z \to (z + 1)(z - 1)^{-1}$ maps the interior (exterior) of the unit circle onto the left (right) half-plane, the result now follows from Proposition 7.4.

The converse to Proposition 1 states:

PROPOSITION 2. *Let* $A(t)$ *be a continuous* $n \times n$ *matrix function defined on an interval* J *and suppose the equation* (1) *has a fundamental matrix* $X(t)$ *satisfying the inequalities*

$$|X(t)\tilde{P}X^{-1}(s)| \le Ke^{-\alpha(t-s)} \quad \text{for} \quad t \ge s ,$$

$$|X(t)(I - \tilde{P})X^{-1}(s)| \le Le^{-\beta(s-t)} \quad \text{for} \quad s \ge t ,$$

where K , L , α , β *are positive constants and*

$$\tilde{P} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} ,$$

For any positive constant $\varepsilon < \min(\alpha, \beta)$ *there exist positive constants* $h = h(K, L, \varepsilon)$ *and* $\delta = \delta(K, L, \varepsilon)$ *such that, if*

$$|A(t_2) - A(t_1)| \le \delta \quad \text{for} \quad |t_2 - t_1| \le h ,$$

then the matrix $A(t)$ *has* k *eigenvalues with real part* $\le -\alpha + \varepsilon$ *and* $n - k$ *eigenvalues with real part* $\ge \beta - \varepsilon$ *for every* t *whose distance from both endpoints of* J *is at least* h .

Proof. We may assume that the norm is Euclidean and that $K = L \ge 1$. Also, by the change of variables $\tilde{x} = e^{(\alpha-\beta)t/2}x$ we can replace both α and β by $\gamma = (\alpha + \beta)/2$ and $A(t)$ by $\tilde{A}(t) = A(t) + \frac{1}{2}(\alpha - \beta)I$.

Suppose $\delta \le \varepsilon/36K^5$ and the interval $\tilde{J} = [u - h, u + h]$ is contained in J . Then we can apply Proposition 5.1 to the autonomous equation

$$\begin{aligned} y' &= \tilde{A}(u)y \\ &= \tilde{A}(t)y + [\tilde{A}(u) - \tilde{A}(t)]y \end{aligned}$$

on the interval $\tilde{J}$. Hence there exists an invertible matrix C such that, in particular,

$$|C\tilde{P}C^{-1}e^{h\tilde{A}(u)}| \le 12K^3e^{-(\gamma-6K^3\delta)h} ,$$

$$|C(I - \tilde{P})C^{-1}e^{-h\tilde{A}(u)}| \le 12K^3e^{-(\gamma-6K^3\delta)h} ,$$

and

$$|C\tilde{P}C^{-1} - X(u)\tilde{P}X^{-1}(u)| \le 144\gamma^{-1}K^6\delta .$$

The last inequality implies

$$|C\tilde{P}C^{-1}| \le 5K .$$

By Lemma 5.1 we can write

$$C\tilde{P}C^{-1} = S\tilde{P}S^{-1} ,$$

where $|S||S^{-1}| \le 10K$. Therefore, putting $T = \exp(hS^{-1}\tilde{A}(u)S)$, we have

$$|\tilde{P}T| \le 120K^4e^{-(\gamma-\varepsilon/6)h} ,$$

$$|(I - \tilde{P})T^{-1}| \le 120K^4e^{-(\gamma-\varepsilon/6)h} .$$

If $h > 0$ is so large that $120K^4 < 2^{-\frac{1}{2}}e^{5\varepsilon h/6}$ then it follows from Lemma 4 that $A(u)$ has k eigenvalues with real part less than $-\alpha + \varepsilon$ and $n - k$ eigenvalues with real part greater than $\beta - \varepsilon$.

It is evident that if the matrix $A(t)$ is continuous on a compact interval J then the equation (1) has an exponential dichotomy on J corresponding to an arbitrary projection P . Thus the concept of exponential dichotomy on a compact interval might appear to be without interest. Nevertheless, Proposition 2 shows that if the interval is sufficiently long relative to the constants of the exponential dichotomy and the coefficient matrix is slowly varying then k , and hence $\tilde{P}$, is uniquely determined.

Finally we give another simple sufficient condition for an exponential dichotomy.

PROPOSITION 3. *Let* $A(t) = (a_{ij}(t))$ *be a bounded, continuous* $n \times n$ *matrix function on the half-line* $\mathbb{R}_+$ *and suppose there exists a constant* $\delta > 0$ *such that*

$$|\mathrm{R}a_{ii}(t)| \ge \delta + \sum_{\substack{j=1 \\ j\ne i}}^{n} |a_{ij}(t)| \tag{7}$$

for all $t \in \mathbb{R}_+$ *and* $i = 1, \ldots, n$. *If* $\mathrm{R}a_{ii}(t) < 0$ *for exactly* k *subscripts* i *then the equation* (1) *has a fundamental matrix* $X(t)$ *satisfying the inequalities*

$$|X(t)\tilde{P}X^{-1}(s)| \le Ke^{-\delta(t-s)} \quad \textit{for} \quad t \ge s ,$$

$$|X(t)(I - \tilde{P})X^{-1}(s)| \le Ke^{-\delta(s-t)} \quad \textit{for} \quad s \ge t ,$$

where K *is a positive constant and*

$$\tilde{P} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} .$$

Proof. We use the norm

$$|\xi| = \sup_i |\xi_i|$$

for an arbitrary vector $\xi = (\xi_i)$. The induced norm of an $n \times n$ matrix $A = (a_{ij})$ is then

$$|A| = \sup_i \sum_{j=1}^{n} |a_{ij}| .$$

The hypothesis (7) implies that the continuous function $\mathcal{R}a_{ii}(t)$ has constant sign. By applying the same permutation to rows and columns we may assume without loss of generality that $\mathcal{R}a_{ii}(t) < 0$ for $i \le k$ and $\mathcal{R}a_{ii}(t) > 0$ for $i > k$.

For any nontrivial solution $x(t)$ of (1) we have

$$\tfrac{1}{2}\, d/dt|x_i|^2 = \mathcal{R}\bar{x}_i x_i{}'$$

$$= \mathcal{R}a_{ii}|x_i|^2 + \mathcal{R}\sum_{j\neq i} a_{ij}\bar{x}_i x_j$$

and hence

$$-|x_i|\sum_{j\neq i}|a_{ij}||x_j| \le \tfrac{1}{2}\, d/dt|x_i|^2 - \mathcal{R}a_{ii}|x_i|^2$$

$$\le |x_i|\sum_{j\neq i}|a_{ij}||x_j| .$$

If $|x(s)| = |x_i(s)|$ for some $s \in \mathbb{R}_+$ it follows that

$$-\sum_{j\neq i}|a_{ij}(s)| \le \tfrac{1}{2}|x_i(s)|^{-2} d/dt|x_i|^2(s) - \mathcal{R}a_{ii}(s)$$

$$\le \sum_{j\neq i}|a_{ij}(s)| .$$

Therefore, by (7), $d/dt|x_i|^2(s)$ is negative if $i \le k$ and positive if $i > k$.

It follows that $|x(t)|$ does not have a local maximum in the interior of $\mathbb{R}_+$. For suppose a local maximum occurred at $t = s$, say, with $|x(s)| = |x_i(s)|$. Then $|x_i(t)|^2$ would also have a local maximum at $t = s$ and hence $d/dt|x_i|^2(s) = 0$, contrary to what has just been proved.

We show next that if $|x(s)| = |x_i(s)|$ for some $i > k$ and some $s \in \mathbb{R}_+$ then

$|x(t)|$ is strictly increasing for $t \geq s$. In fact $|x_i(s)| < |x_i(s+h)|$ for all sufficiently small $h > 0$, and hence $|x(s)| < |x(s+h)|$. If we had $|x(t_1)| \geq |x(t_2)|$, where $s \leq t_1 < t_2$, then $|x(t)|$ would assume its maximum value on the interval $[s, t_2]$ at an interior point, and we have seen that this is impossible.

If $|x(s)| = |x_i(s)|$ for *no* $i > k$ then $|x(s+h)| < |x(s)|$ for all sufficiently small $h > 0$. For let I denote the set of all i such that $|x(s)| = |x_i(s)|$. Then for small $h > 0$

$$|x_i(s+h)| < |x_i(s)| \quad \text{for all} \quad i \in I$$

and

$$|x(s+h)| = |x_i(s+h)| \quad \text{for some} \quad i = i(h) \in I .$$

The set of all s such that $|x(s)| = |x_i(s)|$ for no $i > k$ is open in $\mathbb{R}_+$. Hence by the result of the previous paragraph, there exists a point c such that $|x(t)|$ is strictly decreasing for $t \leq c$ and strictly increasing for $t \geq c$.

We now show that there exists a k-dimensional subspace of solutions $x(t)$ such that $|x(t)|$ is strictly decreasing on $\mathbb{R}_+$. Choose $s_m \in \mathbb{R}_+$ so that $s_m \to \infty$ as $m \to \infty$. Let V_1 be the k-dimensional subspace of all vectors $\xi = (\xi_i)$ such that $\xi_i = 0$ for every $i > k$, and let $X(t)$ be the fundamental matrix of the equation (1) such that $X(0) = I$. Then $U_m = X^{-1}(s_m)V_1$ is a k-dimensional subspace of the underlying vector space V. Let $\xi_m^1, \ldots, \xi_m^k$ be an orthonormal basis for U_m. By the compactness of the unit sphere in V, there exists a sequence of integers $m_\nu \to \infty$ and vectors $\xi^1, \ldots, \xi^k$ such that

$$\xi_{m_\nu}^j \to \xi^j \quad \text{as} \quad \nu \to \infty \; (j = 1, \ldots, k) .$$

Evidently $\xi^1, \ldots, \xi^k$ are mutually orthogonal unit vectors. Hence the solutions $x^1(t), \ldots, x^k(t)$ such that $x^j(0) = \xi^j (j = 1, \ldots, k)$ are linearly independent. Consider any nontrivial linear combination

$$x(t) = \alpha_1 x^1(t) + \ldots + \alpha_k x^k(t) .$$

If $x_m(t)$ is the solution of (1) such that

$$x_m(0) = \alpha_1 \xi_m^1 + \ldots + \alpha_k \xi_m^k$$

then $x_{m_\nu}(t) \to x(t)$ as $\nu \to \infty$ for every $t \in \mathbb{R}_+$. We have $x_m(s_m) \in V_1$, since $x_m(0) \in U_m$. Therefore $|x_m(t)|$ is strictly decreasing for $t \le s_m$, by the definition of V_1. Thus if $t_1, t_2 \in \mathbb{R}_+$ and $t_1 < t_2$ then, for all large ν, $|x_{m_\nu}(t_1)| > |x_{m_\nu}(t_2)|$ and hence $|x(t_1)| \ge |x(t_2)|$. Thus $|x(t)|$ is non-increasing on $\mathbb{R}_+$. It is actually decreasing on $\mathbb{R}_+$, since if $|x(t)|$ were constant over a subinterval it would have a local maximum in the interior of $\mathbb{R}_+$.

There exists also an $(n - k)$-dimensional subspace of solutions $x(t)$ such that $|x(t)|$ is strictly increasing on $\mathbb{R}_+$. In fact, if V_2 is the subspace of all vectors $\xi = (\xi_i)$ such that $\xi_i = 0$ for every $i \le k$, the nontrivial solutions $x(t)$ with $x(0) \in V_2$ have this property.

If $x(t)$ is a solution such that $|x(t)|$ increases on $\mathbb{R}_+$ then, for every $s \in \mathbb{R}_+$, $|x(s)| = |x_i(s)|$ for some $i > k$. Since $d/dt|x_i|^2(s) \ge 2\delta|x_i(s)|^2$, by (7), it follows that

$$\lim_{h \to +0}\left[|x(s+h)|^2 - |x(s)|^2\right]\Big/h \ge \lim_{h \to +0}\left[|x_i(s+h)|^2 - |x_i(s)|^2\right]\Big/h$$

$$\ge 2\delta|x(s)|^2 .$$

Integrating this differential inequality for the continuous function $|x(t)|^2$, we get

$$|x(t)| \ge |x(s)|e^{\delta(t-s)} \quad \text{for} \quad t \ge s .$$

A similar argument shows that if $x(t)$ is a solution such that $|x(t)|$ decreases on $\mathbb{R}_+$ then

$$|x(t)| \le |x(s)|e^{-\delta(t-s)} \quad \text{for} \quad t \ge s .$$

Since the equation (1) has bounded growth, the result now follows immediately.

Proposition 3 remains valid, with only minor changes in the proof, if the half-line $\mathbb{R}_+$ is replaced by the whole line $\mathbb{R}$.

7. DICHOTOMIES AND LYAPUNOV FUNCTIONS

Lyapunov functions are a standard tool in stability theory. In this lecture we will consider their relationship with dichotomies. Since we will be concerned only with linear differential equations it is natural to restrict attention to quadratic Lyapunov functions.

Let $A(t)$ be a continuous matrix function for $t \geq 0$. The Hermitian form $x^*H(t)x$, where $H(t)$ is a bounded and continuously differentiable Hermitian matrix function, will be a Lyapunov function for the linear differential equation

$$x' = A(t)x \tag{1}$$

if its time-derivative along solutions of (1) is negative definite, i.e., if there exists a constant $\eta > 0$ such that

$$x^*(t)[H'(t) + H(t)A(t) + A^*(t)H(t)]x(t) \leq -\eta|x(t)|^2$$

for all solutions $x(t)$ of (1). Here, and in what follows, we use the *Euclidean* norm. By replacing $H(t)$ by $\eta^{-1}H(t)$ we can assume $\eta = 1$. Then, since the solution $x(t)$ is arbitrary, the preceding condition is equivalent to

$$H'(t) + H(t)A(t) + A^*(t)H(t) \leq -I \quad \text{for} \quad t \geq 0 . \tag{2}$$

We are going to show that this is almost equivalent to an exponential dichotomy.

PROPOSITION 1. *If the equation* (1) *has an exponential dichotomy on* $\mathbb{R}_+$ *, then there exists a bounded, continuously differentiable Hermitian matrix function* $H(t)$ *which satisfies* (2).

Proof. Let $X(t)$ be the fundamental matrix for (1) such that $X(0) = I$. By hypothesis there exists a projection P and positive constants K, α such that

$$|X(t)PX^{-1}(s)| \leq Ke^{-\alpha(t-s)} \quad \text{for} \quad t \geq s \geq 0 ,$$

$$|X(t)(I - P)X^{-1}(s)| \leq Ke^{-\alpha(s-t)} \quad \text{for} \quad s \geq t \geq 0 .$$

For convenience of writing put

$$X_1(t, s) = X(t)PX^{-1}(s) ,$$

$$X_2(t, s) = X(t)(I - P)X^{-1}(s) ,$$

and let $H(t)$ be the matrix function defined by

$$\tfrac{1}{2}H(t) = \int_t^{\infty} X_1{}^*(s, t)X_1(s, t)ds - X_2{}^*(0, t)X_2(0, t)$$

$$- \int_0^t X_2{}^*(s, t)X_2(s, t)ds .$$

Then $H(t)$ is Hermitian and

$$\tfrac{1}{2}|H(t)| \le K^2\int_t^{\infty} e^{-2\alpha(s-t)}ds + K^2e^{-2\alpha t} + K^2\int_0^t e^{-2\alpha(t-s)}ds$$

$$\le K^2(1 + \alpha^{-1}) .$$

Moreover $H(t)$ is continuously differentiable and

$$H'(t) = -H(t)A(t) - A^*(t)H(t) - 2X_1{}^*(t, t)X_1(t, t)$$

$$- 2X_2{}^*(t, t)X_2(t, t) .$$

But $X_1(t, t) = X(t)PX^{-1}(t)$ is a projection and $X_2(t, t) = I - X_1(t, t)$. Thus it only remains to show that if Q is a projection then

$$R = Q^*Q + (I - Q^*)(I - Q) \ge \tfrac{1}{2}I .$$

Any vector ξ can be uniquely represented in the form $\xi = \xi_1 + \xi_2$, where ξ_1 is in the range and ξ_2 in the nullspace of Q . Then

$$\xi_1{}^*R\xi_1 = \xi_1{}^*\xi_1 , \quad \xi_2{}^*R\xi_2 = \xi_2{}^*\xi_2 ,$$

$$\xi_2{}^*R\xi_1 = 0 .$$

Hence

$$\xi^*R\xi = \xi_1{}^*\xi_1 + \xi_2{}^*\xi_2 \ge \tfrac{1}{2}\xi^*\xi .$$

This completes the proof.

It may be noted that if $A(t)$ is bounded then $H'(t)$ is also bounded. Proposition 1 admits the following converse:

PROPOSITION 2. *Suppose* (1) *has bounded growth. If there exists a bounded, continuously differentiable Hermitian matrix function* $H(t)$ *which satisfies* (2), *then the equation* (1) *has an exponential dichotomy on* $\mathbb{R}_+$.

Proof. By hypothesis there exists a constant $\rho > 0$ such that

$$|H(t)| \le \rho \quad \text{for} \quad t \ge 0$$

and a constant $C \ge 1$ such that every solution $x(t)$ of (1) satisfies

$$|x(t)| \le C|x(s)| \quad \text{for} \quad 0 \le s \le t \le s + e\rho \ .$$

(The interval length $e\rho$ is chosen to give a good exponent in the exponential dichotomy). Assume $|x(0)| = 1$ and put

$$V(t) = x^*(t)H(t)x(t) \ .$$

Then

$$\begin{aligned} V'(t) &= x^*(t)[H'(t) + H(t)A(t) + A^*(t)H(t)]x(t) \\ &\le -|x(t)|^2 \\ &< 0 \ . \end{aligned}$$

Thus $V(t)$ is a strictly decreasing function.

Suppose first that $V(t) \ge 0$ for all $t \ge 0$. Then

$$\int_0^\infty |x(u)|^2 du < \infty \ ,$$

since for any $t \ge 0$

$$\begin{aligned} -\rho = -\rho|x(0)|^2 &\le -V(0) \\ &\le V(t) - V(0) \\ &\le -\int_0^t |x(u)|^2 du \ . \end{aligned}$$

It follows that

$$\underline{\lim}_{t\to\infty} |x(t)| = 0 \ .$$

Let t_m be the least value such that $|x(t_m)| = e^{-m/2}$. Then $0 = t_0 < t_1 < \ldots$ and, by the same argument as above,

$$\begin{aligned} -\rho|x(t_m)|^2 &\le -\int_{t_m}^{t_{m+1}} |x(u)|^2 du \\ &< -(t_{m+1} - t_m)|x(t_{m+1})|^2 \ . \end{aligned}$$

Since $|x(t_m)|^2 = e|x(t_{m+1})|^2$ it follows that $t_{m+1} - t_m \le e\rho$. Suppose $0 \le s \le t < \infty$. If $t_m \le s < t_{m+1}$ and $t_n \le t < t_{n+1} (0 \le m \le n)$, then

$$\begin{aligned} |x(t)| &\le C|x(t_n)| = eCe^{-(n+1-m)/2}|x(t_{m+1})| \\ &< eCe^{-(n+1-m)/2}|x(s)| \\ &\le eCe^{-\alpha(t-s)}|x(s)| , \end{aligned}$$

where $\alpha = (2e\rho)^{-1}$.

Suppose next that $V(t^*) < 0$ for some $t^* \ge 0$. Then for $t \ge t^*$

$$-\rho|x(t)|^2 \le V(t) \le V(t^*)$$

and hence

$$V'(t) \le \rho^{-1}V(t^*) < 0 .$$

Therefore $V(t) \to -\infty$ as $t \to \infty$, which implies $|x(t)| \to \infty$.

Let V be the underlying vector space, let V_1 be the subspace consisting of the initial values of all bounded solutions of (1), and let V_2 be any fixed subspace supplementary to V_1 . For any unit vector $\xi \in V_2$, let $x(t) = x(t, \xi)$ be the solution which takes the value ξ at $t = 0$. We will show that $V(t) = V(t, \xi) \to -\infty$ *uniformly* in ξ as $t \to \infty$. In fact, otherwise there exists a sequence of unit vectors $\xi_\nu \in V_2$ and a sequence $t_\nu \to \infty$ such that

$$V(t_\nu, \xi_\nu) \ge \mu \text{ for all } \nu .$$

By the compactness of the unit sphere in V_2 we may suppose that $\xi_\nu \to \xi$. Since $V(t, \xi) \to -\infty$ as $t \to \infty$ we have $V(t', \xi) < \mu$ for some t' . Then $V(t', \xi_\nu) < \mu$ for $\nu \ge \nu'$, and hence $V(t, \xi_\nu) < \mu$ for $t \ge t'$ and $\nu \ge \nu'$, which is a contradiction. Thus there exists a positive number T such that $V(T, \xi) \le -\rho$ for every unit vector $\xi \in V_2$. Then $|x(t, \xi)| > 1$ for every unit vector $\xi \in V_2$ and all $t > T$. Moreover, we can choose $N \ge 1$ so that $|x(T, \xi)| \le N|x(t, \xi)|$ for every unit vector $\xi \in V_2$ and $0 \le t \le T$.

Consider again a particular solution $x(t) = x(t, \xi)$. Since $|x(t)| \to \infty$ there exists a greatest value t_m such that $|x(t_m)| = e^{m/2}$. Then $0 \le t_0 < t_1 < \dots$ and $t_0 \le T$. Suppose $t \ge T$. If $t_m \le t < t_{m+1}$ then

$$\begin{aligned} -\rho|x(t_{m+1})|^2 &\le V(t_{m+1}) \\ &\le V(t_{m+1}) - V(t) \\ &\le -\int_t^{t_{m+1}} |x(u)|^2 du \\ &\le -(t_{m+1} - t)|x(t_m)|^2 . \end{aligned}$$

Since $|x(t_{m+1})|^2 = e|x(t_m)|^2$ it follows that $t_{m+1} - t_m \le e\rho$ if $t_m \ge T$ and $t_{m+1} - T \le e\rho$ if $t_m < T$. In either case

$$|x(t)| \le CN|x(t_m)| .$$

Suppose $T \le t \le s < \infty$. If $t_m \le t < t_{m+1}$ and $t_n \le s < t_{n+1} (0 \le m \le n)$, then

$$\begin{aligned} |x(t)| \le CN|x(t_m)| &= e^{\frac{1}{2}}CNe^{-(n+1-m)/2}|x(t_n)| \\ &\le e^{\frac{1}{2}}CNe^{-(n+1-m)/2}|x(s)| \\ &\le e^{\frac{1}{2}}CNe^{-\alpha(s-t)}|x(s)| , \end{aligned}$$

where $\alpha = (2e\rho)^{-1}$ as before. Since (1) has bounded growth, it follows that it has an exponential dichotomy on the subinterval $[T, \infty)$, and hence also on the half-line $\mathbb{R}_+$.

We show next, by an example, that the hypothesis of bounded growth cannot be omitted in Proposition 2. Let $\phi(t)$ be a real-valued continuously differentiable function such that $\phi(t) \ge 1$ for all $t \ge 0$, $\int_0^\infty \{\phi^2(t) - 1\}dt < \infty$, $\phi(t) = o(e^t)$ for $t \to \infty$, and $\phi(n)/\phi(n - 2^{-n}) \to \infty$. Such a function can easily be constructed explicitly. The differential equation

$$x' = [\phi'(t)/\phi(t) - 1]x$$

has the solutions $x(t) = x(0)e^{-t}\phi(t)$ and hence is asymptotically stable, but not uniformly stable. On the other hand, if we set

$$h(t) = \{e^{2t}/\phi^2(t)\}\int_t^\infty e^{-2s}\phi^2(s)ds$$

then $h(t)$ is continuously differentiable and

$$h'(t) + 2\{\phi'(t)/\phi(t) - 1\}h(t) = -1 .$$

Moreover $h(t)$ is bounded, since

$$0 \le h(t) = \{e^{2t}/\phi^2(t)\}\int_t^\infty e^{-2s}ds$$

$$+ \{e^{2t}/\phi^2(t)\}\int_t^\infty e^{-2s}\{\phi^2(s) - 1\}ds$$

$$\le \tfrac{1}{2} + \int_0^\infty \{\phi^2(s) - 1\}ds \ .$$

The preceding results can be made more precise in the special case of autonomous equations. Thus we now consider the equation

$$x' = Ax \ ,$$

where A is a constant matrix. We show first that the matrix $H(t)$ of Proposition 1 can also be chosen constant. The argument differs only slightly from the previous one.

PROPOSITION 3. *If the matrix* A *has no pure imaginary eigenvalues, then there exists a Hermitian matrix* H *such that*

$$HA + A^*H \le -I \ .$$

Proof. Let P_+ and P_- be the spectral projections of A for the right and left half-planes respectively. Then

$$Y = -\int_0^\infty e^{-tA^*}P_+{}^*P_+e^{-tA}dt$$

is defined and

$$YA + A^*Y = \int_0^\infty \frac{d}{dt}(e^{-tA^*}P_+{}^*P_+e^{-tA})dt$$

$$= -P_+{}^*P_+ \ .$$

Similarly

$$Z = \int_0^\infty e^{tA^*}P_-{}^*P_-e^{tA}dt$$

is defined and

$$ZA + A^*Z = -P_-{}^*P_- \ .$$

The matrix $H = 2Y + 2Z$ is Hermitian and

$$HA + A^*H = -2P_+{}^*P_+ - 2P_-{}^*P_- \le -I \ .$$

Proposition 2 can be sharpened for autonomous equations by relaxing the requirement that the time derivative of the Lyapunov function be negative definite. For any matrix C, the ordered pair (A, C) is said to be *controllable* if

$$\zeta^*A^kC = 0 \quad \text{for all} \quad k \ge 0$$

implies that the vector $\zeta = 0$. By the Cayley-Hamilton theorem we need only consider k such that $0 \le k < n$.

PROPOSITION 4. *Suppose there exists a Hermitian matrix* H *such that*

$$HA + A^*H = -C , \tag{3}$$

where $C = C^* \ge 0$. *Then* A *has no pure imaginary eigenvalues and* H *is non-singular if and only if* (A^*, C) *is controllable. In this case the number of positive (negative) eigenvalues of* H *is equal to the number of eigenvalues of* A *with negative (positive) real part.*

Proof. Suppose first that A has a pure imaginary eigenvalue $i\omega$, with corresponding eigenvector ζ . Then, by (3),

$$-\zeta^*C\zeta = i\omega\zeta^*H\zeta - i\omega\zeta^*H\zeta = 0 .$$

Since $C \ge 0$, this implies that $C\zeta = 0$. Therefore

$$CA^k\zeta = (i\omega)^k C\zeta = 0 \quad \text{for all } k \ge 0 .$$

Thus (A^*, C) is not controllable.

Suppose next that A has no pure imaginary eigenvalues. Let γ be the simple closed curve in the left half-plane consisting of a segment of the imaginary axis and the left half of the circle $|z| = r$, where $r > |A|$. Then the spectral projections P_+ and P_- of A , for the right and left half-planes, are given by

$$P_- = \frac{1}{2\pi i}\int_\gamma (zI - A)^{-1}dz ,$$

$$P_+ = \frac{1}{2\pi i}\int_\gamma (zI + A)^{-1}dz .$$

Here the expression for P_+ has been obtained by the change of variable $z \to -z$ from the more usual expression by a contour integral over the curve symmetric to γ . If we set

$$G = H - P_-^*H - HP_-$$

then

$$G = \frac{1}{2\pi i}\int_\gamma \{(zI + A^*)^{-1}H - H(zI - A)^{-1}\}dz$$

$$= \frac{1}{2\pi i}\int_\gamma (zI + A^*)^{-1}C(zI - A)^{-1}dz .$$

On the circle $|z| = r$ the integrand is $O(r^{-2})$. Thus, letting $r \to \infty$ we obtain

$$G = -\frac{1}{2\pi}\int_{-\infty}^{\infty} (A - i\omega I)^{*-1}C(A - i\omega I)^{-1}d\omega .$$

Consequently $G \leq 0$. Moreover $G\xi = 0$ if and only if $C(A - i\omega I)^{-1}\xi = 0$ for all real ω , and hence if and only if $CA^k\xi = 0$ for all $k \geq 0$. Therefore $G < 0$ if and only if (A^*, C) is controllable.

For any vector $\xi \in V_+ = P_+V$ we have $P_-\xi = 0$ and hence

$$\xi^*G\xi = \xi^*H\xi \ .$$

On the other hand, for any vector $\xi \in V_- = P_-V$ we have $P_-\xi = \xi$ and hence

$$\xi^*G\xi = -\xi^*H\xi \ .$$

In general, if $\xi = \xi_+ + \xi_-$, where $\xi_+ \in V_+$ and $\xi_- \in V_-$, then

$$\xi^*G\xi = \xi_+{}^*H\xi_+ - \xi_-{}^*H\xi_- \ .$$

Therefore H is non-positive on V_+ and non-negative on V_- , and $G\xi = 0$ if and only if $H\xi_+ = H\xi_- = 0$. Thus if G is singular, then H is also singular.

If G is non-singular, then H is negative definite on V_+ and positive definite on V_- . It follows from the minimax principle that the number of positive (negative) eigenvalues of H is at least equal to $\dim V_-$ ($\dim V_+$) . Since the total number of eigenvalues is $\dim V_- + \dim V_+$, we must actually have equality and H must be non-singular. This completes the proof.

8. EQUATIONS ON $\mathbb{R}$ AND ALMOST PERIODIC EQUATIONS

In Lecture 3 we considered, for the half-line $\mathbb{R}_+$, the existence of bounded solutions of the inhomogeneous equation

$$y' = A(t)y + f(t) \tag{1}$$

for, say, every bounded continuous function f . Corresponding results for the half-line $\mathbb{R}_-$ may be obtained by the change of variable $t \to -t$. We will now show that the same question for the whole line $\mathbb{R}$ can be answered in terms of the results for the two half-lines.

PROPOSITION 1. *The inhomogeneous linear differential equation* (1) *has at least one solution bounded on* $\mathbb{R}$ *for every function* $f \in B(\mathbb{R})$, *where* B *denotes any one of the Banach spaces* C , M , L , *if and only if the following three conditions are satisfied:*

(i) *the equation* (1) *has at least one solution bounded on* $\mathbb{R}_+$ *for every function* $f \in B(\mathbb{R}_+)$,

(ii) *the equation* (1) *has at least one solution bounded on* $\mathbb{R}_-$ *for every function* $f \in B(\mathbb{R}_-)$,

(iii) *every solution of the corresponding homogeneous equation*

$$x' = A(t)x \tag{2}$$

is the sum of a solution which is bounded on $\mathbb{R}_+$ *and a solution which is bounded on* $\mathbb{R}_-$.

Proof. Suppose first that the three conditions are all satisfied. By (i) and (ii), for any function $f \in B(\mathbb{R})$ the equation (1) has solutions $y_+(t)$, $y_-(t)$ which are bounded on $\mathbb{R}_+$, $\mathbb{R}_-$ respectively. By (iii) there exist solutions $x_+(t)$, $x_-(t)$ of the homogeneous equation (2) which are bounded on $\mathbb{R}_+$, $\mathbb{R}_-$ respectively and such that

$$x_+(0) - x_-(0) = y_+(0) - y_-(0) .$$

Then

$$y(t) = y_+(t) - x_+(t) , \ \tilde{y}(t) = y_-(t) - x_-(t)$$

are solutions of (1) which are bounded on $\mathbb{R}_+$, $\mathbb{R}_-$ respectively and $y(0) = \tilde{y}(0)$. Hence $y(t) = \tilde{y}(t)$ is a solution of (1) which is bounded on $\mathbb{R}$.

Conversely, suppose (1) has a solution bounded on $\mathbb{R}$ for every function $f \in B(\mathbb{R})$. Then (i) and (ii) are clearly satisfied, since every function in $B(\mathbb{R}_+)$, or $B(\mathbb{R}_-)$, is the restriction to $\mathbb{R}_+$, or $\mathbb{R}_-$, of a function in $B(\mathbb{R})$.

Take $f(t) = \phi(t)x(t)$, where $x(t)$ is any fixed solution of (2) and

$$\phi(t) = \begin{cases} 1 - |t| & \text{for } |t| \le 1 , \\ 0 & \text{for } |t| > 1 . \end{cases}$$

Then $f \in B(\mathbb{R})$ and

$$y(t) = \int_0^t \phi(u)du \, x(t)$$

is a solution of the corresponding equation (1). Since

$$\int_0^1 \phi(u)du = \int_{-1}^0 \phi(u)du = \tfrac{1}{2} ,$$

we have

$$y(t) = \tfrac{1}{2}x(t) \quad \text{for} \quad t \ge 1 , \ y(t) = -\tfrac{1}{2}x(t) \quad \text{for} \quad t \le -1 .$$

By hypothesis the equation (1) has a solution $\tilde{y}(t)$ which is bounded on $\mathbb{R}$. Then

$$z_+(t) = \tilde{y}(t) - y(t) + \tfrac{1}{2}x(t) ,$$

$$z_-(t) = \tilde{y}(t) - y(t) - \tfrac{1}{2}x(t) ,$$

are solutions of the homogeneous equation (2) which are bounded on $\mathbb{R}_+$, $\mathbb{R}_-$ respectively. Moreover, since $y(0) = 0$,

$$z_+(0) = \tilde{y}(0) + \tfrac{1}{2}x(0) ,$$

$$z_-(0) = \tilde{y}(0) - \tfrac{1}{2}x(0) .$$

Hence $x(0) = z_+(0) - z_-(0)$ and $x(t) = z_+(t) - z_-(t)$. This completes the proof.

By combining Proposition 1 with Propositions 3.1-3.3 we can obtain more explicit conditions. We will consider only one important special case.

PROPOSITION 2. *Suppose* (2) *has bounded growth. Then the inhomogeneous equation* (1) *has a unique solution bounded on* $\mathbb{R}$ *for every continuous function* f *which is bounded on* $\mathbb{R}$ *if and only if the homogeneous equation* (2) *has an exponential dichotomy on* $\mathbb{R}$.

Proof. Let $X(t)$ be the fundamental matrix for (2) such that $X(0) = I$ and suppose there exist a projection P and positive constants K , α such that

$$
\begin{aligned}
|X(t)PX^{-1}(s)| &\le Ke^{-\alpha(t-s)} \quad \text{for} \quad t \ge s \, , \\
|X(t)(I - P)X^{-1}(s)| &\le Ke^{-\alpha(s-t)} \quad \text{for} \quad s \ge t \, .
\end{aligned}
\tag{3}
$$

Then for any bounded continuous function f the corresponding inhomogeneous equation (1) has the bounded solution

$$
y(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)f(s)ds - \int_{t}^{\infty} X(t)(I - P)X^{-1}(s)ds \, . \tag{4}
$$

Moreover, this bounded solution is unique, since the homogeneous equation (2) has no nontrivial bounded solution. Here we have not used the hypothesis of bounded growth.

To prove the necessity of the condition we use Proposition 1. Since (2) has bounded growth, (i) and (ii) imply the existence of exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$, by Proposition 3.3. (Proposition 3.3 remains valid if the hypothesis of bounded growth is replaced by a hypothesis of bounded decay). Let P_+ , P_- be corresponding projections. Then P_+V and $(I - P_-)V$ span V , by (iii), and have only the zero vector in common, because (2) has no nontrivial bounded solution. Hence there exists a projection P with range P_+V and nullspace $(I - P_-)V$, and (2) has an exponential dichotomy on $\mathbb{R}$ with projection P .

A matrix function $A(t)$ is said to be *almost periodic* if every sequence $\{h_\nu\}$ of real numbers contains a subsequence $\{k_\nu\}$ such that the translates $A(t + k_\nu)$ converge uniformly on $\mathbb{R}$ as $\nu \to \infty$. This implies that $A(t)$ is bounded and uniformly continuous on $\mathbb{R}$.

When the coefficient matrix $A(t)$ of (2) is almost periodic the preceding results can be strengthened. We first prove

LEMMA 1. *Let* $A(t)$ *be a continuous matrix function on* $\mathbb{R}$ *and suppose the equation* (2) *has an exponential dichotomy* (3) *on* $\mathbb{R}_+$. *If, for some sequence* $h_\nu \to \infty$, $A(t + h_\nu) \to B(t)$ *uniformly on compact subintervals of* $\mathbb{R}$ *then* $X(h_\nu)PX^{-1}(h_\nu) \to Q$ *and the equation*

$$y' = B(t)y \tag{5}$$

has an exponential dichotomy on $\mathbb{R}$ *with projection* Q *and the same constants* K, α .

Proof. The translated equation

$$x' = A(t + h_\nu)x$$

has the fundamental matrix $X_\nu(t) = X(t + h_\nu)X^{-1}(h_\nu)$ and

$$|X_\nu(t)P_\nu X_\nu^{-1}(s)| \le Ke^{-\alpha(t-s)} \quad \text{for} \quad t \ge s \ge -h_\nu ,$$

$$|X_\nu(t)(I - P_\nu)X^{-1}(s)| \le Ke^{-\alpha(s-t)} \quad \text{for} \quad s \ge t \ge -h_\nu ,$$

where $P_\nu = X(h_\nu)PX^{-1}(h_\nu)$. Since $|P_\nu| \le K$, by restricting attention to a subsequence we can assume that $P_\nu \to Q$, where Q is a projection. Since $X_\nu(t) \to Y(t)$ for every t , where $Y(t)$ is the fundamental matrix of (5) such that $Y(0) = I$, it follows that

$$|Y(t)QY^{-1}(s)| \le Ke^{-\alpha(t-s)} \quad \text{for} \quad -\infty < s \le t < \infty ,$$

$$|Y(t)(I - Q)Y^{-1}(s)| \le Ke^{-\alpha(s-t)} \quad \text{for} \quad -\infty < t \le s < \infty .$$

Since the projection corresponding to an exponential dichotomy on $\mathbb{R}$ is uniquely determined it follows that $P_\nu \to Q$ without restriction to a subsequence.

PROPOSITION 3. *Suppose* $A(t)$ *is an almost periodic matrix function. Then the following statements are equivalent:*

(i) *the homogeneous equation* (2) *has an exponential dichotomy on* $\mathbb{R}_+$,

(ii) *the homogeneous equation* (2) *has an exponential dichotomy on* $\mathbb{R}$,

(iii) *the inhomogeneous equation* (1) *has a solution bounded on* $\mathbb{R}$ *for every almost periodic function* $f(t)$,

(iv) *the inhomogeneous equation* (1) *has a solution bounded on* $\mathbb{R}_+$ *for every almost periodic function* $f(t)$.

Proof. (i) implies (ii): This follows at once from Lemma 1, since there exists a sequence $h_\nu \to \infty$ such that $A(t + h_\nu) \to A(t)$ uniformly on $\mathbb{R}$.

(ii) implies (iii): This was established in the first part of the proof of Proposition 2.

(iii) implies (iv): This is trivial.

(iv) implies (i): Since $A(t)$ is bounded it is sufficient, by Proposition 3.3, to show that (1) has a solution bounded on $\mathbb{R}_+$ for every bounded continuous function $f(t)$ if it has a solution bounded on $\mathbb{R}_+$ for every almost periodic function $f(t)$.

The set A of all functions $f(t)$ which are restrictions to the half-line $\mathbb{R}_+$ of almost periodic functions on $\mathbb{R}$ is a closed subspace of the Banach space $C(\mathbb{R}_+)$. In fact, for any almost periodic function $f(t)$,

$$\sup_{-\infty<t<\infty} |f(t)| = \overline{\lim_{t\to\infty}} |f(t)| \ ;$$

hence any fundamental sequence of almost periodic functions on $\mathbb{R}_+$ is also a fundamental sequence on $\mathbb{R}$, and its limit in $C(\mathbb{R})$ is an almost periodic function.

The proof of Proposition 3.4 shows that there exists a constant $r = r_A > 0$ such that, for any function $f \in A$, the equation (1) has a solution $y(t)$ which is bounded on $\mathbb{R}_+$ and satisfies

$$\|y\| \le r\|f\| \ ,$$

where $\|y\| = \sup_{t\ge 0}|y(t)|$.

Let $g(t)$ be any function in $C(\mathbb{R}_+)$ and let T be any positive number. For any ω such that $0 < \omega < 2\pi/T$ we can find a continuous function $f(t)$ with period $2\pi/\omega > T$ which coincides with $g(t)$ on $[0, T]$ and satisfies $\|f\| \le \|g\|$. Then the equation (1) has a bounded solution $y(t)$ with $\|y\| \le r\|g\|$.

Now give T a sequence of values $T_\nu \to \infty$. We obtain corresponding sequences of continuous periodic functions $f_\nu(t)$ and bounded solutions $y_\nu(t)$ with $\|y_\nu\| \le r\|g\|$. Thus the sequence $\{y_\nu(0)\}$ is bounded. By restricting attention to a subsequence we can assume that $y_\nu(0) \to \eta$. Since $f_\nu(t) \to g(t)$ uniformly on compact intervals, it follows that $y_\nu(t) \to y(t)$ for every t , where $y(t)$ is the solution of the equation

$$y' = A(t)y + g(t)$$

such that $y(0) = \eta$. Evidently $y(t)$ is bounded and $\|y\| \leq r\|g\|$. This completes the proof.

The argument in the last part of the proof establishes rather more than has been asserted. Let S be any module, i.e., an additive subgroup of $\mathbb{R}$, which is *dense* in $\mathbb{R}$. Then the set $A(S)$ of all almost periodic functions $f(t)$ with Fourier series $\sum c_k e^{i\lambda_k t}$, where $\lambda_k \in S$ for all k , is a closed subspace of A . The proof shows that in (iv) it is sufficient to assume the existence of a solution bounded on $\mathbb{R}_+$ for every $f \in A(S)$. However, the hypothesis that S is dense in $\mathbb{R}$ cannot be omitted. For example, the scalar equation

$$y' = \tfrac{1}{2}iy + f(t)$$

has a solution of period 2π , and hence a bounded solution, for every continuous function $f(t)$ of period 2π .

It should be noted also that in (iii) the solution bounded on $\mathbb{R}$ is unique, by (ii). Furthermore it is almost periodic and its frequency module is contained in the joint frequency module of $A(t)$ and $f(t)$. This may be established by a similar argument to that used to prove our next result.

PROPOSITION 4. *Suppose* $A(t)$ *is an almost periodic matrix function and the equation* (2) *has an exponential dichotomy* (3), *with fundamental matrix* $X(t)$ *and projection* P . *Then* $X(t)PX^{-1}(t)$ *is almost periodic and its frequency module is contained in the frequency module of* $A(t)$.

Proof. Let $\{h_\nu\}$ be any sequence of real numbers such that $A(t + h_\nu)$ converges uniformly on $\mathbb{R}$, with limit $B(t)$ say. If we put $U(t) = X(t)PX^{-1}(t)$ then, by the normality properties of almost periodic functions, it is sufficient to show that $U(t + h_\nu)$ converges uniformly on $\mathbb{R}$.

By Lemma 1 and its proof, as $\nu \to \infty$

$$\begin{aligned} U(t + h_\nu) &= X_\nu(t)U(h_\nu)X_\nu^{-1}(t) \\ &\to Y(t)QY^{-1}(t) = V(t) \text{ , say,} \end{aligned}$$

where Q is a projection and $Y(t)$ is the fundamental matrix of the equation (5) such that $Y(0) = I$. Moreover the convergence is uniform on compact intervals. If it were not uniform on $\mathbb{R}$ there would exist a sequence $\{t_\nu\}$ of real numbers and a subsequence $\{k_\nu\}$ of $\{h_\nu\}$ such that, for all ν ,

$$|U(t_\nu + k_\nu) - V(t_\nu)| \geq \gamma > 0 .$$

By restricting attention to a further subsequence we may suppose that $A(t + t_\nu + k_\nu)$ converges uniformly on $\mathbb{R}$, with limit $C(t)$ say. Then also $B(t + t_\nu) \to C(t)$ uniformly on $\mathbb{R}$. By what we have already proved, $U(t + t_\nu + k_\nu)$ converges, uniformly on compact intervals, to $Z(t)\tilde{P}Z^{-1}(t)$, where $Z(t)$ is the fundamental matrix of the equation

$$z' = C(t)z$$

such that $Z(0) = I$ and $\tilde{P}$ is a uniquely determined projection. Similarly, $V(t + t_\nu)$ converges to $Z(t)\tilde{P}Z^{-1}(t)$. Therefore

$$|U(t_\nu + k_\nu) - V(t_\nu)| \to 0 ,$$

which is a contradiction.

9. DICHOTOMIES AND THE HULL OF AN EQUATION

Let $\tilde{A}(t)$ be an $n \times n$ matrix function which is bounded and uniformly continuous on the whole line $\mathbb{R}$. Then, by Ascoli's theorem, any sequence $\{h_\nu\}$ of real numbers contains a subsequence $\{k_\nu\}$ such that the translates $\tilde{A}(t + k_\nu)$ converge locally uniformly, i.e. uniformly on every compact interval. The limit function $A(t)$ is again bounded and uniformly continuous. The set $\mathcal{A}$ of all such limits $A(t)$ is called the *hull* of $\tilde{A}(t)$.

The set $\mathcal{A}$ is translation invariant, i.e., if $A(t) \in \mathcal{A}$ then also $A(t + h) \in \mathcal{A}$ for any real number h. Moreover $A(t + h) \to A(t)$ locally uniformly (and even uniformly on $\mathbb{R}$) as $h \to 0$. The set $\mathcal{A}$ is also compact, i.e., any sequence of elements of $\mathcal{A}$ contains a subsequence which converges locally uniformly to an element of $\mathcal{A}$. If $A_\nu(t) \to A(t)$ locally uniformly, $\xi_\nu \to \xi$, and $s_\nu \to s$ then $x_\nu(s_\nu) \to x(s)$, where $x_\nu(t)$ is the solution of the differential equation

$$x' = A_\nu(t)x$$

such that $x_\nu(0) = \xi_\nu$ and $x(t)$ is the solution of the differential equation

$$x' = A(t)x$$

such that $x(0) = \xi$. It is just these properties of the hull that we will actually use.

We propose to study the *family* of differential equations

$$x' = A(t)x\ , \quad A \in \mathcal{A}\ . \tag{1}$$

We first prove

LEMMA 1. *The equations* (1) *have uniformly bounded growth and decay; that is, there exists a constant* $C > 0$ *such that every solution* $x(t)$ *of an equation* (1) *satisfies*

$$|x(t)| \le C|x(s)| \quad \textit{for} \quad |t - s| \le 1 .$$

Proof. Otherwise, since $\mathcal{A}$ is translation invariant, there exists a sequence of solutions $x_\nu(t)$ of equations

$$x' = A_\nu(t)x \ , \ A_\nu \in \mathcal{A} \tag{1$_\nu$}$$

with $|x_\nu(0)| = 1$, and a sequence of real numbers $\{s_\nu\}$ with $-1 \le s_\nu \le 1$, such that $|x_\nu(s_\nu)| \to \infty$. By restricting attention to a subsequence we may suppose that $A_\nu(t) \to A(t)$ locally uniformly, $x_\nu(0) \to \xi$ where $|\xi| = 1$, and $s_\nu \to s$ where $-1 \le s \le 1$. Then $x_\nu(s_\nu) \to x(s)$, where $x(t)$ is the solution of the equation (1) such that $x(0) = \xi$. Thus we have a contradiction.

Until further notice we now make the following fundamental assumption:

(H) *No equation* (1) *has a nontrivial solution bounded on* $\mathbb{R}$.

With its aid we first prove

LEMMA 2. *If* (H) *holds, and if* $0 < \theta < 1$, *there exists a constant* $T > 0$ *such that every solution* $x(t)$ *of an equation* (1) *satisfies*

$$|x(t)| \le \theta \sup_{|u-t| \le T} |x(u)| \quad \textit{for} \quad -\infty < t < \infty .$$

Proof. Otherwise, since $\mathcal{A}$ is translation invariant, there is no $T > 0$ such that

$$\theta^{-1} \le \sup_{|u| \le T} |x(u)|$$

for every solution $x(t)$ of an equation (1) with $|x(0)| = 1$. Thus there exists a sequence $t_\nu \to \infty$, and a sequence of solutions $x_\nu(t)$ of equations $(1)_\nu$ with $|x_\nu(0)| = 1$, such that

$$\sup_{|t| \le t_\nu} |x_\nu(t)| < \theta^{-1} .$$

By restricting attention to a subsequence we may suppose that $x_\nu(0) \to \xi$, where $|\xi| = 1$, and $A_\nu(t) \to A(t)$ locally uniformly, where $A \in \mathcal{A}$. Then $x_\nu(t) \to x(t)$ for every real t, where $x(t)$ is the solution of (1) such that $x(0) = \xi$. For each fixed t we have $|x_\nu(t)| < \theta^{-1}$ for all sufficiently large ν, and hence $|x(t)| \le \theta^{-1}$. Therefore, by the hypothesis (H), $x(t) \equiv 0$. Thus $\xi = 0$, which is a contradiction.

It follows from Lemma 2, by Proposition 2.1 and its proof, that there exist positive constants K, α such that for every solution $x(t)$ of an equation (1) which is bounded on $\mathbb{R}_+$,

$$|x(t)| \le Ke^{-\alpha(t-s)}|x(s)| \quad \text{for} \quad -\infty < s \le t < \infty .$$

Moreover each equation (1) has an exponential dichotomy on $\mathbb{R}_+$. Similarly, for every solution $x(t)$ of an equation (1) which is bounded on $\mathbb{R}_-$,

$$|x(t)| \le Ke^{-\alpha(s-t)}|x(s)| \quad \text{for} \quad -\infty < t \le s < \infty ,$$

and each equation (1) has an exponential dichotomy on $\mathbb{R}_-$.

If, for a given equation (1), each solution is the sum of a solution bounded on $\mathbb{R}_+$ and a solution bounded on $\mathbb{R}_-$ then, by the hypothesis (H), this equation has an exponential dichotomy on $\mathbb{R}$ and the constants of the exponential dichotomy are independent of the particular equation, since K and α are independent and the equations have uniformly bounded growth and decay. It follows that the norm of the projection, corresponding to the fundamental matrix which takes the value I at $t = 0$, is also bounded independently of the particular equation. If the equations $(1)_\nu$ have exponential dichotomies on $\mathbb{R}$ with projections P_ν and if $A_\nu(t) \to A(t)$ locally uniformly, then $P_\nu \to P$, where P is again a projection, and the equation (1) has an exponential dichotomy on $\mathbb{R}$ with projection P. In fact the bounded sequence $\{P_\nu\}$ has a convergent subsequence. If P is its limit then the limit equation (1) has an exponential dichotomy on $\mathbb{R}$ with projection P, since the approximating equations have exponential dichotomies on $\mathbb{R}$ with constants independent of ν. Thus P is uniquely determined, and so the whole sequence $\{P_\nu\}$ converges to P.

Consider now an arbitrary equation (1) and suppose $A(t + h_\nu) \to A_\omega(t)$ locally uniformly for some sequence $h_\nu \to \infty$. If P_+ is a projection corresponding to the exponential dichotomy of the equation (1) on the half-line $\mathbb{R}_+$ then, by Lemma 8.1, the equation

$$x' = A_\omega(t)x$$

has an exponential dichotomy on $\mathbb{R}$ with projection of the same rank as P_+. Similarly, suppose $A(t + k_\nu) \to A_\alpha(t)$ locally uniformly for some sequence $k_\nu \to -\infty$. If P_- is a projection corresponding to the exponential dichotomy of the equation (1) on the half-line $\mathbb{R}_-$ then the equation

$$x' = A_\alpha(t)x$$

has an exponential dichotomy on $\mathbb{R}$ with projection of the same rank as P_- .

The preceding results establish, in particular,

PROPOSITION 1. *Suppose that no equation* (1) *in the hull* A *of the equation*

$$x' = \tilde{A}(t)x \tag{2}$$

has a nontrivial solution bounded on $\mathbb{R}$.

Then each equation (1) *in the* α- *or* ω- *limit set of* (2) *has an exponential dichotomy on* $\mathbb{R}$. *Moreover there exist positive constants* K , α *such that, if* $X(t, A)$ *is the fundamental matrix of* (1) *for which* $X(0, A) = I$,

$$|X(t, A)P(A)X^{-1}(s, A)| \le Ke^{-\alpha(t-s)} \quad \textit{for} \quad -\infty < s \le t < \infty ,$$

$$|X(t, A)(I - P(A))X^{-1}(s, A)| \le Ke^{-\alpha(s-t)} \quad \textit{for} \quad -\infty < t \le s < \infty ,$$

where $P(A)$ *is a uniquely determined projection depending continuously on* A *(with the topology of uniform convergence on compact intervals).*

We also have

PROPOSITION 2. *The following statements are equivalent:*

(i) *no equation* (1) *in the hull* A *of* (2) *has a nontrivial solution bounded on* $\mathbb{R}$,

(ii) *the equation* (2) *has an exponential dichotomy on each of the half-lines* $\mathbb{R}_+$, $\mathbb{R}_-$ *with corresponding projections* P_+ , P_- *such that*

$$P_+P_- = P_-P_+ = P_+ .$$

Proof. (i) implies (ii): Only the conditions on the projections remain to be proved. These say that the nullspace of P_- is contained in the nullspace of P_+ and the range of P_+ is contained in the range of P_- . But, since the range of P_+ and the nullspace of P_- intersect trivially, we can choose P_+ so that the first condition is satisfied and then P_- so that the second condition is also satisfied.

(ii) implies (i): Let $\tilde{A}(t + h_\nu) \to A(t)$ locally uniformly. We may suppose that $h_\nu \to h$, where $-\infty \le h \le \infty$. If h is finite then $A(t) = \tilde{A}(t + h)$ and the equation (1) has no nontrivial bounded solution because the range of P_+ and the nullspace of

P_- intersect trivially. If $h = \pm\infty$ then, by Lemma 8.1, the equation (1) has an exponential dichotomy on $\mathbb{R}$ and the same conclusion holds.

The conditions of Proposition 2 do not guarantee the existence of an exponential dichotomy on the whole line $\mathbb{R}$. Consider, for example, the scalar equation

$$x' = (\tanh t)x ,$$

whose solutions are given by $x(t) = x(0)\cosh t$. The hull A of the bounded and uniformly continuous function $\tanh t$ consists of all functions $\tanh(t+h)$, where $-\infty < h < \infty$, and the two constants -1, 1. Hence no equation in the hull has a nontrivial solution which is bounded on $\mathbb{R}$. On the other hand, the given equation has no nontrivial solution which is bounded on either $\mathbb{R}_+$ or $\mathbb{R}_-$, and therefore does not have an exponential dichotomy on $\mathbb{R}$.

This example can easily be generalised. Let $\tilde{A}(t)$ be a continuous matrix function for $t \in \mathbb{R}$ such that the limits

$$A_+ = \lim_{t\to+\infty} \tilde{A}(t) \ , \ A_- = \lim_{t\to-\infty} \tilde{A}(t)$$

exist and have no pure imaginary eigenvalues. If A_+ and A_- do not have the same number of eigenvalues with negative real part, then the equation (2) does not have an exponential dichotomy on $\mathbb{R}$. But if, in addition, the equation (2) has no non-trivial solution $\tilde{x}(t)$ such that $\tilde{x}(t) \to 0$ for both $t\to+\infty$ and $t\to-\infty$, then no equation (1) in the hull of (2) has a nontrivial solution bounded on $\mathbb{R}$.

A bounded and uniformly continuous matrix function $\tilde{A}(t)$ is said to be *recurrent* if, for every $A(t)$ in the hull of $\tilde{A}(t)$, also $\tilde{A}(t)$ is in the hull of $A(t)$. For example, any almost periodic function is recurrent.

The preceding problem does not arise if $\tilde{A}(t)$ is recurrent. For then $\tilde{A}(t)$ is both an α- and an ω-limit of itself, and from Proposition 1 we obtain

PROPOSITION 3. *Suppose* $\tilde{A}(t)$ *is recurrent. Then the equation* (2) *has an exponential dichotomy on* $\mathbb{R}$ *if and only if no equation* (1) *in the hull* A of $\tilde{A}(t)$ *has a nontrivial solution bounded on* $\mathbb{R}$.

In the almost periodic case this provides another equivalent to the statements of Proposition 8.3. In particular, statement (iii) shows that in a classical theorem of Favard [1] the hypothesis that the inhomogeneous equation has a bounded solution is redundant. It may be noted also that for almost periodic $\tilde{A}(t)$ the definition of the hull A is not altered if locally uniform convergence is replaced by uniform convergence on $\mathbb{R}$.

Proposition 3 characterises recurrent equations with exponential dichotomies. We now turn our attention to recurrent equations with ordinary dichotomies. Thus we drop the hypothesis (H) .

We first prove

LEMMA 3. *Suppose* $\tilde{A}(t)$ *is continuous on* $\mathbb{R}$ *and the equation* (2) *has an* m*-dimensional subspace of solutions such that, for each nontrivial solution* $\tilde{x}(t)$ *in this subspace,*

$$0 < \inf_{t\in\mathbb{R}_+} |\tilde{x}(t)| \le \sup_{t\in\mathbb{R}_+} |\tilde{x}(t)| < \infty .$$

Suppose also that $\tilde{A}(t + h_\nu) \to A(t)$ *locally uniformly for some sequence* $h_\nu \to \infty$.

Then the ω*-limit equation* (1) *has an* m*-dimensional subspace of solutions such that, for each nontrivial solution* $x(t)$ *in this subspace,*

$$0 < \inf_{t\in\mathbb{R}} |x(t)| \le \sup_{t\in\mathbb{R}} |x(t)| < \infty .$$

Proof. Let $\tilde{x}_1(t), \ldots, \tilde{x}_m(t)$ be a basis for the given subspace of solutions of (2). By restricting attention to a subsequence we may suppose that $\tilde{x}_j(h_\nu) \to \xi_j$ $(j = 1, \ldots, m)$. Then $\tilde{x}_j(t + h_\nu) \to x_j(t)$ for every real t , where $x_j(t)$ is the solution of the equation (1) such that $x_j(0) = \xi_j$ $(j = 1, \ldots, m)$. Evidently the solutions $x_1(t), \ldots, x_m(t)$ are bounded on $\mathbb{R}$. They are also linearly independent. For suppose

$$c_1\xi_1 + \ldots + c_m\xi_m = 0$$

and put

$$\tilde{x}(t) = c_1\tilde{x}_1(t) + \ldots + c_m\tilde{x}_m(t) .$$

Since $\tilde{x}(h_\nu) \to 0$ we must have $\tilde{x}(t) \equiv 0$. Hence $c_1 = \ldots = c_m = 0$.

Let

$$x(t) = a_1x_1(t) + \ldots + a_mx_m(t)$$

be a nontrivial linear combination of $x_1(t), \ldots, x_m(t)$. Then if

$$\tilde{x}(t) = a_1\tilde{x}_1(t) + \ldots + a_m\tilde{x}_m(t)$$

there is a $\delta > 0$ such that $|\tilde{x}(t)| \ge \delta$ for every $t \ge 0$. Since $\tilde{x}(t + h_\nu) \to x(t)$ it follows that $|x(t)| \ge \delta$ for every real t .

PROPOSITION 4. *Suppose* $\tilde{A}(t)$ *is recurrent and the equation* (2) *has an ordinary dichotomy on* $\mathbb{R}_+$. *Then there exist positive constants* K , α *such that, if* $X(t, A)$

is the fundamental matrix of an equation (1) *in the hull of* (2) *with* $X(0, A) = I$,

$$|X(t, A)P_0(A)X^{-1}(s, A)| \le K \qquad \textit{for} \quad -\infty < s , t < \infty ,$$

$$|X(t, A)P_+(A)X^{-1}(s, A)| \le Ke^{-\alpha(t-s)} \quad \textit{for} \quad -\infty < s \le t < \infty ,$$

$$|X(t, A)P_-(A)X^{-1}(s, A)| \le Ke^{-\alpha(s-t)} \quad \textit{for} \quad -\infty < t \le s < \infty ,$$

where $P_0(A)$, $P_+(A)$, $P_-(A)$ *are uniquely determined supplementary projections depending continuously on* A *(with the topology of uniform convergence on compact intervals)* .

Proof. Let $\tilde{P}$ be a projection whose range is the subspace of initial values of solutions of (2) which are bounded on $\mathbb{R}_+$. The equation (2) has a dichotomy on $\mathbb{R}_+$ with projection $\tilde{P}$, by Proposition 2.2. Since every equation (1) in the hull of (2) is an ω-limit of (2), the argument in the first part of the proof of Lemma 8.1 shows that each equation (1) has a dichotomy on $\mathbb{R}$ with projection P of the same rank as $\tilde{P}$. Thus we could have started with (1) instead of (2). It follows that the range of P is exactly the subspace V_1 of initial values of solutions of (1) which are bounded on $\mathbb{R}_+$. Thus every solution $x(t)$ of an equation (1) which is bounded on $\mathbb{R}_+$ satisfies an inequality

$$|x(t)| \le L|x(s)| \quad \text{for} \quad -\infty < s \le t < \infty .$$

Hence either $x(t) \to 0$ as $t \to \infty$ or $\inf|x(t)| > 0$ for $t \in \mathbb{R}_+$. Moreover, if $x(t)$ is nontrivial, either $|x(t)| \to \infty$ as $t \to -\infty$ or $x(t)$ is bounded on $\mathbb{R}_-$ and $\inf|x(t)| > 0$ for $t \in \mathbb{R}_-$.

Let $V_+ \subset V_1$ be the subspace of initial values of solutions of (1) which tend to zero as $t \to \infty$ and let W be any subspace of V_1 supplementary to V_+ . It follows from Lemma 3 that the equation (1) has a subspace of solutions with the same dimension as W such that each nontrivial solution in this subspace is bounded and bounded away from zero on $\mathbb{R}$. If V_0 is the corresponding subspace of initial values then

$$V_1 = V_0 \dotplus V_+ .$$

Similarly, let V_2 be the subspace of initial values of solutions of (1) which are bounded on $\mathbb{R}_-$ and $V_- \subset V_2$ the subspace of initial values of solutions of (1) which tend to zero as $t \to -\infty$. Then in the same way we can write

$$V_2 = V_0' \dotplus V_- ,$$

where V_0' is of the same nature as V_0. Since V_1 and V_- intersect trivially, $\dim V_0 \le \dim V_0'$, and since V_2 and V_+ intersect trivially, $\dim V_0' \le \dim V_0$. Hence $\dim V_0' = \dim V_0$ and we can take $V_0' = V_0$.

Finally, since (1) has a dichotomy on $\mathbb{R}$, every solution is the sum of a solution bounded on $\mathbb{R}_+$ and a solution bounded on $\mathbb{R}_-$. Therefore the entire vector space V admits the direct decomposition

$$V = V_0 \dotplus V_+ \dotplus V_- .$$

The preceding argument shows also that the dimensions of the three components are the same for all equations (1). Any solution which is bounded on $\mathbb{R}$ has its initial value in both $V_0 \dotplus V_+$ and $V_0 \dotplus V_-$, and hence in V_0. Thus V_0 is just the subspace of initial values of solutions which are bounded on $\mathbb{R}$, and every nontrivial solution bounded on $\mathbb{R}$ is bounded away from zero. Furthermore, if $x(t)$ is a nontrivial solution with initial value in V_+ (V_-) then $|x(t)| \to \infty$ as $t \to -\infty$ $(+\infty)$.

Consider, in particular, the equation (2) and let $\tilde{V}_j = V_j(\tilde{A})$ $(j = 0, +, -)$. There exist uniquely determined supplementary projections $\tilde{P}_0$, $\tilde{P}_+$, $\tilde{P}_-$ with ranges $\tilde{V}_0$, $\tilde{V}_+$, $\tilde{V}_-$ respectively. By Proposition 2.2, the equation (2) has a dichotomy with projection $\tilde{P}_0 + \tilde{P}_+$ on both $\mathbb{R}_+$ and $\mathbb{R}_-$, and hence also on $\mathbb{R}$. Similarly it has a dichotomy with projection $\tilde{P}_+$ on $\mathbb{R}$. It follows that there exists a constant $K > 0$ such that

$$|\tilde{X}(t)\tilde{P}_+\tilde{X}^{-1}(s)| \le K \quad \text{for} \quad -\infty < s \le t < \infty ,$$

$$|\tilde{X}(t)\tilde{P}_-\tilde{X}^{-1}(s)| \le K \quad \text{for} \quad -\infty < t \le s < \infty ,$$

$$|\tilde{X}(t)\tilde{P}_0\tilde{X}^{-1}(s)| \le K \quad \text{for} \quad -\infty < s\ ,\ t < \infty ,$$

where $\tilde{X}(t) = X(t, \tilde{A})$.

Suppose $\tilde{A}(t + h_\nu) \to A(t)$ locally uniformly. Then by passing to a subsequence we see that there exist supplementary projections $P_+(A)$, $P_-(A)$, $P_0(A)$ with the same ranks as $\tilde{P}_+$, $\tilde{P}_-$, $\tilde{P}_0$ such that

$$|X(t, A)P_+(A)X^{-1}(s, A)| \le K \quad \text{for} \quad -\infty < s \le t < \infty ,$$

$$|X(t, A)P_-(A)X^{-1}(s, A)| \le K \quad \text{for} \quad -\infty < t \le s < \infty ,$$

$$|X(t, A)P_0(A)X^{-1}(s, A)| \le K \quad \text{for} \quad -\infty < s , t < \infty .$$

Any nontrivial solution $x_0(t)$ of the equation (1) with initial value in $P_0(A)V$ is bounded, and bounded away from zero, on $\mathbb{R}$. Since the dimensions are equal, it follows that $V_0 = P_0(A)V$. Any nontrivial solution $x_+(t)$ starting from $P_+(A)V$ is bounded on $\mathbb{R}_+$. Therefore, by Lemma 3, there is a solution $x_0(t)$ starting from $P_0(A)V$ such that

$$\inf_{t \in \mathbb{R}_+} |x_0(t) + x_+(t)| = 0 .$$

Since

$$|x_+(t)| \le K|x_0(s) + x_+(s)| \quad \text{for} \quad t \ge s$$

this implies that $x_+(t) \to 0$ as $t \to \infty$. Since the dimensions are equal, if follows that $V_+ = P_+(A)V$. Similarly, $V_- = P_-(A)V$. Thus the projections $P_0(A)$, $P_+(A)$, $P_-(A)$ are uniquely determined and, without passing to a subsequence,

$$\tilde{X}(h_\nu)\tilde{P}_j\tilde{X}^{-1}(h_\nu) \to P_j(A) \quad (j = 0, +, -) .$$

We are now in a position to apply our previous arguments under the hypothesis (H) to the subspace $\tilde{V}_- \dot{+} \tilde{V}_+$. First of all we have the analogue of Lemma 2: if $0 < \theta < 1$ there exists $T > 0$ such that every solution $\tilde{x}(t)$ of (2) with $\tilde{x}(0) \in \tilde{V}_- \dot{+} \tilde{V}_+$ satisfies

$$|\tilde{x}(t)| \le \theta \sup_{|u-t| \le T} |\tilde{x}(u)| .$$

For otherwise there exists a sequence $T_\nu \to \infty$, a real sequence $\{t_\nu\}$, and a sequence of solutions $\tilde{x}_\nu(t)$ of (2) with $\tilde{x}_\nu(0) \in \tilde{V}_- \dot{+} \tilde{V}_+$ and $|\tilde{x}_\nu(t_\nu)| = 1$, such that

$$\sup_{|u-t_\nu| \le T_\nu} |\tilde{x}_\nu(u)| < \theta^{-1} .$$

Moreover we may assume that $\tilde{A}(t + t_\nu) \to A(t)$ locally uniformly and $\tilde{x}_\nu(t_\nu) \to \xi$, where $|\xi| = 1$. Then $\tilde{x}_\nu(t + t_\nu) \to x(t)$ for every t; where $x(t)$ is the solution of the equation (1) such that $x(0) = \xi$. Hence $|x(t)| \le \theta^{-1}$ for $-\infty < t < \infty$. On the other hand, since $\tilde{P}_0\tilde{x}_\nu(0) = 0$ also

$$P_0(A)\xi = \lim_{\nu\to\infty} \tilde{X}(t_\nu)\tilde{P}_0\tilde{X}^{-1}(t_\nu) \cdot \tilde{X}(t_\nu)\tilde{x}_\nu(0) = 0 .$$

Thus we have a contradiction.

It follows that, 'with respect to the subspace $\tilde{V}_- \dot{+} \tilde{V}_+'$, the equation (2) has an exponential dichotomy on $\mathbb{R}$, the corresponding projections being necessarily $\tilde{P}_-$, $\tilde{P}_+$. The remaining assertions of Proposition 4 now follow without difficulty.

Proposition 4 provides an answer to a problem of Hahn. It shows that for any almost periodic, or even recurrent, linear differential system uniform stability and asymptotic stability on $\mathbb{R}_+$ together imply uniform asymptotic stability on $\mathbb{R}$. In fact we then have $P_- = P_0 = 0$. Thus if $a(t)$ is any almost periodic function with mean value zero such that

$$\int_0^t a(s)ds \to -\infty \quad \text{as} \quad t \to \infty$$

then the scalar differential equation

$$x' = a(t)x$$

is asymptotically stable but not uniformly stable. For example, one can take

$$a(t) = - \sum_{n=1}^{\infty} n^{-3/2} \sin \pi t/n .$$

Proposition 4 may also be used to discuss the *orbital stability* of a quasi-periodic solution of an autonomous nonlinear system. Let $g(u)$ be a continuously differentiable function from $\mathbb{R}^m$ into $\mathbb{R}^n$ with period 1 in each coordinate u_j , and let $\omega_1, \ldots, \omega_m$ be real numbers linearly independent over the rationals. We suppose that

$$x_0(t) = g(\omega_1 t, \ldots, \omega_m t)$$

is a solution of the autonomous differential equation

$$x' = \phi(x) , \tag{3}$$

where ϕ is continuously differentiable on the range of g .

Then all functions

$$x_h(t) = g(\omega_1 t+h_1, \ldots, \omega_m t+h_m) \qquad (h \in \mathbb{R}^m)$$

in the hull of $x_0(t)$ are also solutions of (3), and the m partial derivatives

$g_{u_j}(\omega_1 t, \ldots, \omega_m t)$ are quasi-periodic solutions of the variational equation

$$y' = \phi_x[x_0(t)]y \ . \tag{4}$$

We assume that the variational equation (4) is uniformly stable on $\mathbb{R}_+$ and that these m solutions form a basis for the solutions which are bounded on $\mathbb{R}$.

The hull of (4) consists of all equations

$$y' = \phi_x[x_h(t)]y \ .$$

Let $Y_h(t)$ be the fundamental matrix of this equation such that $Y_h(0) = I$. By Proposition 4 there exist positive constants K , α such that

$$\begin{aligned} |Y_h(t)P_hY_h^{-1}(s)| &\le Ke^{-\alpha(t-s)} \quad \text{for } -\infty < s \le t < \infty , \\ |Y_h(t)(I - P_h)Y_h^{-1}(s)| &\le K \qquad\qquad \text{for } -\infty < s , t < \infty , \end{aligned} \tag{5}$$

where P_h is a uniquely determined projection which depends continuously on h . By hypothesis, the vectors $g_{u_j}(0)$ $(j = 1, \ldots, m)$ are a basis for the null space of P_0 .

If in (3) we make the change of variable $x = z + x_h(t)$ we obtain the equation

$$z' = \phi_x[x_h(t)]z + f_h(t, z) ,$$

where

$$f_h(t, z) = \phi[z + x_h(t)] - \phi[x_h(t)] - \phi_x[x_h(t)]z \ .$$

Thus f_h is a quasi-periodic function of t for each fixed z , $f_h(t, 0) = 0$, and

$$|f_h(t, z_1) - f_h(t, z_2)| \le \varepsilon|z_1 - z_2| \quad \text{for all } t \text{ and } h$$

if $|z_1| \le \delta$, $|z_2| \le \delta$, where $\delta = \delta(\varepsilon) > 0$.

Choose γ so that $0 < \gamma < \alpha$ and fix $\varepsilon > 0$ so small that

$$\theta = \varepsilon K[\gamma^{-1} + (\alpha-\gamma)^{-1}] < 1 \ .$$

Let ξ be any vector in the range of P_0 such that $|\xi| < (1 - \theta)K^{-1}\delta$, and let $z(t)$ be any continuous function from $\mathbb{R}_+$ into $\mathbb{R}^n$ such that

$$\|z\| = \sup_{t\geq 0} e^{\gamma t}|z(t)| \leq \delta .$$

Consider the integral operator

$$Tz(t) = Y_h(t)P_h\xi + \int_o^t Y_h(t)P_hY_h^{-1}(s)f_h[s, z(s)]ds$$

$$- \int_t^\infty Y_h(t)(I - P_h)Y_h^{-1}(s)f_h[s, z(s)]ds .$$

Using (5), it is easily verified that

$$\|Tz\| \leq K|\xi| + \theta\|z\| < \delta$$

and

$$\|Tz_1 - Tz_2\| \leq \theta\|z_1 - z_2\| .$$

It follows, by the contraction principle, that for any given vector ξ the operator T has a unique fixed point $z_h(t, \xi)$ in the ball $\|z\| \leq \delta$, and

$$\|z_h(t, \xi)\| \leq (1 - \theta)^{-1}K|\xi| .$$

Moreover, for $t = 0$,

$$z_h(0, \xi) = P_h\xi - \int_o^\infty (I - P_h)Y_h^{-1}(s)f_h[s, z_h(s, \xi)]ds$$

and hence, as $|\xi| \to 0$,

$$z_h(0, \xi) = P_h\xi + o(|\xi|) \quad \text{uniformly in } h .$$

The fixed point $z_h(t, \xi)$ is a solution of the differential equation

$$z' = \phi_x[x_h(t)]z + f_h(t, z)$$

and thus

$$x_h(t, \xi) = z_h(t, \xi) + x_h(t)$$

is a solution of the original equation (3). Thus for each h we have an $(n-m)$-dimensional family of solutions of (3) which converge to $x_h(t)$ exponentially as $t \to \infty$.

We wish to show that the function $(h, \xi) \to x_h(0, \xi)$ maps a neighbourhood of the origin onto a neighbourhood of the point $x_o(0)$. For $|h| + |\xi| \to 0$ we have

$$z_h(0, \xi) = \xi + o(|\xi|) ,$$

$$x_h(0) = g(0) + g_u(0)h + o(|h|) ,$$

and hence

$$x_h(0, \xi) = x_o(0) + g_u(0)h + \xi + o(|h| + |\xi|) .$$

The linear map $(h, \xi) \to g_u(0)h + \xi$ is invertible, since $g_u(0)h$ and ξ run through the nullspace and range of the projection P_o . Therefore the assertion follows from the topological inverse function theorem.

Consequently, for each solution $x(t)$ of (3) which passes sufficiently near the m-dimensional torus $g(u)$, on which the given quasi-periodic solution $x_o(t)$ is dense, there exists a vector $h = (h_1, \ldots, h_m)$ such that the difference

$$x(t) - g(\omega_1 t + h_1, \ldots, \omega_m t + h_m)$$

tends to zero exponentially as $t \to \infty$. This generalises a standard result on the orbital stability of periodic solutions.

APPENDIX: THE METHOD OF PERRON

We describe here an alternative approach to the problems of Lecture 3, mentioned in the Notes, and give an application which does not seem amenable to our previous functional-analytic approach.

Let $A(t)$ be a continuous $n \times n$ matrix function on the half-line $\mathbb{R}_+$ and let $X(t)$ be a fundamental matrix for the linear differential equation

$$x' = A(t)x \ . \tag{1}$$

By Gram-Schmidt orthogonalisation of the columns of $X(t)$, starting with the last column, we obtain a unitary matrix $U(t)$ and a lower triangular matrix $Y(t)$ such that

$$X(t) = U(t)Y(t) \ .$$

Moreover, $U(t)$ and $Y(t)$ are uniquely determined if we require the elements in the main diagonal of $Y(t)$ to be real and positive. Since $X(t)$ is continuously differentiable, it is easily seen that then $U(t)$ and $Y(t)$ are also.

The change of variables $x = U(t)y$ replaces the equation (1) by

$$y' = B(t)y \ , \tag{2}$$

where $B = U^{-1}AU - U^{-1}U'$. Since $Y(t)$ is a fundamental matrix of the transformed equation, $B(t) = Y'(t)Y^{-1}(t)$ is lower triangular with real main diagonal. Moreover, since U is unitary,

$$(UU^*)' = U'U^* + UU^{*\prime} = 0$$

and hence

$$U(B + B^*)U^* = A + A^* \ .$$

Therefore

$$\|B + B^*\| = \|A + A^*\| \ .$$

For any $n \times n$ matrix $C = (c_{jk})$ it is easily shown that

$$n^{-1} \sum_{j,k} |c_{jk}|^2 \leq \|C\|^2 \leq \sum_{j,k} |c_{jk}|^2 .$$

Since B is triangular and $b_{kk} = b^*_{kk}$, it follows that

$$\begin{aligned} \|B\|^2 &\leq \sum_{j,k} |b_{jk}|^2 \leq \frac{1}{2} \sum_{j,k} |b_{jk} + b^*_{jk}|^2 \\ &\leq \tfrac{1}{2}n\|B + B^*\|^2 = \tfrac{1}{2}n\|A + A^*\|^2 \\ &\leq 2n\|A\|^2 . \end{aligned}$$

Thus

$$\|B(t)\| \leq (2n)^{\frac{1}{2}}\|A(t)\| .$$

The inhomogeneous equation

$$z' = A(t)z + f(t) \tag{3}$$

will have a bounded solution $z(t)$ if and only if the transformed equation

$$w' = B(t)w + g(t) , \tag{4}$$

where $g(t) = U^{-1}(t)f(t)$, has a bounded solution $w(t) = U^{-1}(t)z(t)$. Hence the equation (3) will have a bounded solution for every bounded continuous function $f(t)$ if and only if the equation (4) has a bounded solution for every bounded continuous function $g(t)$. If $A(t)$ is real then $B(t)$ is also real, and if $A(t)$ is bounded then $B(t)$ is also bounded. It follows that to prove Proposition 3.3, for the case of a bounded coefficient matrix, we can assume that the coefficient matrix is also triangular and use induction on the dimension n . We will not carry out the details here, but obtain instead a new result.

PROPOSITION 1. *Let* $B(t)$ *be a continuous* $n \times n$ *matrix function on* $\mathbb{R}_+$ *, which is lower triangular with real main diagonal and bounded off-diagonal elements.*

Then there exists a constant $\delta = \delta(B) > 0$ *such that the inhomogeneous linear differential equation* (4) *has a bounded solution for every bounded continuous function* $g(t)$ *if it has a bounded solution for* $g(t) = g_1(t). \ldots, g_n(t)$, where

$$g_k(t) = \begin{pmatrix} g_{1k}(t) \\ \vdots \\ g_{nk}(t) \end{pmatrix}$$

is a continuous function such that, for all $t \geq 0$,

$$\mathcal{R}g_{kk}(t) \geq 1 \ , \ |g_{jk}(t)| \leq \delta \ (j = 1, \ldots, k-1) \ .$$

Proof. Suppose first that $n = 1$. Then

$$y(t) = \exp \int_0^t B(u)du$$

is a positive solution of the corresponding homogeneous equation (2). The equation (4) with $g(t) = \mathcal{R}g_1(t)$ has a real bounded solution $w_1(t)$. Moreover $w_1(t)$ has the form

$$w_1(t) = y(t)\{w_1(0) + \int_0^t \mathcal{R}g_1(u)/y(u)\,du\} \ .$$

The expression between the braces has a finite or infinite limit as $t \to \infty$. If this limit is zero then

$$w_1(t) = -y(t)\int_t^\infty \mathcal{R}g_1(u)/y(u)\,du \ .$$

If it is non-zero then there is a constant $c > 0$ such that

$$y(t) \leq c|w_1(t)| \quad \text{for all large} \quad t \ ,$$

and hence also

$$y(t)\int_0^t \mathcal{R}g_1(u)/y(u)\,du$$

is a bounded solution of (4). Since $\mathcal{R}g_1(u) \geq 1$ it follows that either

$$\int_t^\infty y(t)/y(u)\,du$$

or

$$\int_0^t y(t)/y(u)\,du$$

is bounded. Hence for any bounded continuous function $g(t)$ the equation (4) has the bounded solution

$$w(t) = -y(t)\int_t^\infty g(u)/y(u)\,du$$

or

$$w(t) = y(t)\int_0^t g(u)/y(u)\,du \ .$$

Suppose next that $n > 1$ and that the result holds in lower dimensions. We write

$$B(t) = \begin{pmatrix} \tilde{B}(t) & 0 \\ b(t) & \beta(t) \end{pmatrix} ,$$

where $\tilde{B}(t)$ is an $(n - 1) \times (n - 1)$ matrix and $\beta(t)$ is a scalar. Let $\tilde{\delta}$ be the positive constant corresponding to $\tilde{B}(t)$ by the induction hypothesis and suppose $\delta \le \tilde{\delta}$. Then the inhomogeneous equation

$$\tilde{w}' = \tilde{B}(t)\tilde{w} + \tilde{g}(t)$$

has a bounded solution $\tilde{w}(t)$ for every bounded continuous function $\tilde{g}(t)$. Moreover, by Proposition 3.4 and Lemma 3.2,

$$||\tilde{w}(t)|| \le eNe^{-r^{-1}t}||\tilde{w}(0)|| + r \sup ||\tilde{g}(t)|| ,$$

where r and N are positive constants depending only on $\tilde{B}(t)$. By hypothesis, for $g(t) = g_n(t)$ the equation (4) has a bounded solution

$$w_n(t) = \begin{pmatrix} \tilde{w}_n(t) \\ \omega(t) \end{pmatrix} .$$

Thus $\omega(t)$ is a bounded solution of the scalar equation

$$\omega' = \beta(t)\omega + b(t)\tilde{w}_n(t) + g_{nn}(t) .$$

We can choose $\delta > 0$ so small that

$$r(n - 1)^{\frac{1}{2}} \delta \sup_{t\ge 0} ||b(t)|| \le \frac{1}{4}$$

and then $t_0 > 0$ so large that

$$||b(t)\tilde{w}_n(t)|| \le \frac{1}{2} \quad \text{for} \quad t \ge t_0 .$$

Then, by what we have proved for $n = 1$, the scalar equation

$$\omega' = \beta(t)\omega + \gamma(t)$$

has a bounded solution for every bounded continuous scalar function $\gamma(t)$. It now follows from the induction hypothesis that the equation (4) has a bounded solution for every bounded continuous function $g(t)$.

PROPOSITION 2. *Let* $A(t)$ *be a bounded continuous real* $n \times n$ *matrix function on* $\mathbb{R}_+$ *and let* $x_1(t), \ldots, x_n(t)$ *be a fundamental system of solutions of the linear differential equation* (1).

Then there exists a constant $\rho > 0$ *such that* (1) *has an exponential dichotomy if the corresponding inhomogeneous equation* (3) *has a bounded solution for* $f(t) = f_1(t), \ldots, f_n(t)$ *, where* $f_k(t)$ *is a continuous function whose distance from the subspace spanned by* $x_{k+1}(t), \ldots, x_n(t)$ *is at least* 1 *and whose distance from the subspace spanned by* $x_k(t), \ldots, x_n(t)$ *is less than* ρ *, for all* $t \geq 0$.

Proof. Let $X(t)$ be the fundamental matrix of (1) with columns $x_1(t), \ldots, x_n(t)$ and form the corresponding system (2) with triangular coefficient matrix. By Proposition 3.3 and our previous remarks, it is sufficient to show that the inhomogeneous equation (4) has a bounded solution for every bounded continuous function $g(t)$. We know that (4) has a bounded solution for $g(t) = g_k(t) = U^{-1}(t)f_k(t)$ $(k = 1, \ldots, n)$. If we denote the columns of $U(t)$ by $u_1(t), \ldots, u_n(t)$ then $u_k(t)$ is a unit vector in the subspace spanned by $x_k(t), \ldots, x_n(t)$ which is orthogonal to $x_{k+1}(t), \ldots, x_n(t)$. Since

$$f_k(t) = g_{1k}(t)u_1(t) + \ldots + g_{nk}(t)u_n(t)$$

we have

$$g_{1k}^2(t) + \ldots + g_{k-1,k}^2(t) \leq \rho ,$$

$$g_{1k}^2(t) + \ldots + g_{kk}^2(t) \geq 1 .$$

The result now follows from Proposition 1.

In the complex case we can prove in the same way

PROPOSITION 3. *Let* $A(t)$ *be a bounded continuous* $n \times n$ *matrix function on* $\mathbb{R}_+$. *Then there exists a constant* $\eta > 0$ *and continuously differentiable functions* $u_1(t), \ldots, u_n(t)$ *with* $\|u_k(t)\| \equiv 1$ *such that the equation* (1) *has an exponential dichotomy if the corresponding inhomogeneous equation* (3) *has a bounded solution for* $f(t) = f_1(t), \ldots, f_n(t)$ *, where* $f_k(t)$ *is a continuous function satisfying* $\|f_k(t) - u_k(t)\| < \eta$ *for all* $t \geq 0$.

In particular, when the coefficient matrix is bounded the inhomogeneous equation (3) has a bounded solution for every $f \in C$ if it has a bounded solution for every f in some dense subset of C .

NOTES

Lecture 1. For the variation of constants formula, Gronwall's inequality, and further information about stability and nonlinear problems see, e.g., Coppel [1]. An eigenvalue is said to be semisimple if the corresponding elementary divisors are linear, i.e., all corresponding blocks in the Jordan canonical form are 1×1. The example of the failure of the eigenvalue characterisation in the non-autonomous case comes from Hoppensteadt [1]. The theory of functions of a matrix is discussed in Dunford and Schwartz [1] and in Hille and Phillips [1]. For the inequality (4) see Gel'fand and Shilov [1], p.65. A less precise version of Proposition 4 is given in Coppel [1], p.117. For Proposition 6 see Coppel [3]. The main properties of the 'logarithmic norm' $\mu(A)$ are described in Coppel [1], pp.41 and 58.

Lecture 2. The terms 'exponential dichotomy' and 'ordinary dichotomy' are due to Massera and Schäffer. For definitions of the angle between two subspaces see Massera and Schäffer [3], Chapter 1, and Daleckiĭ and Kreĭn [1], Chapter 4. Proposition 1 is due to Massera and Schäffer [2], although it was suggested by earlier work of Krasovskiĭ [1]. An indirect proof of Proposition 2 is given in Massera and Schäffer [3], Chapters 4 and 6.

Lecture 3. Questions of the type studied here were first considered by Perron [1], although Bohl [1] had come very close. Perron assumed the coefficient matrix $A(t)$ bounded and showed that by a change of variables it could be supposed triangular, thus permitting induction on the dimension. Maĭzel' [1] first established Proposition 3, using Perron's method. An interesting application of this method, recently given by Rahimberdiev [1], is discussed in the Appendix.

For the case in which all solutions of (1) are bounded Kreĭn [1] and Bellman [1] used instead the uniform boundedness theorem of functional analysis. Massera and Schäffer [1] used the closed graph theorem, as in Proposition 4, to establish both Propositions 1 and 3, and later also Proposition 2. Their theory is set out in full generality in Massera and Schäffer [3]. For the closed graph theorem itself see, e.g., Dunford and Schwartz [1] or Hille and Phillips [1].

Lecture 4. The roughness of exponential dichotomies is treated in Massera and Schäffer [3], Chapter 7, and in Daleckiĭ and Kreĭn [1], Chapter 4. For the successive approximations argument used in the proof of Lemma 1 cf. Coppel [1], p.13.

Lecture 5. The term 'kinematic similarity' comes from Markus [1]. Lemmas 1-3 are due to Coppel [2]. However, the use in Lemma 2 of the fundamental matrix of (4) follows Daleckiĭ and Kreĭn [1]. The smoothness properties of the positive square root of a positive Hermitian matrix follow at once from its representation by a contour integral. If, in Proposition 1, the original coefficient matrix $A(t)$ commutes with P for every t then the change of variables $x = T(t)z$ is superfluous and one obtains much sharper estimates for the various constants. Propositions 2 and 3 are proved in Coppel [3].

Lecture 6. Schäffer [1] shows that in Lemma 1 one can take $k_n = (en)^{\frac{1}{2}}$ and conjectures that the best possible value is $k_n = 2$ for $n > 1$. Proposition 1 is an extension, due to Chang and Coppel [1], of a result in Coppel [2]. The important differential equation (4) was discovered independently by Daleckiĭ and Kreĭn [1]$^{+}$ and Kato [1]. Daleckiĭ and Kreĭn [1] give an extension of Proposition 1 to Hilbert space. An approach which permits an extension to Banach space is contained in a recent article of Žikov [1]. Proposition 2, which is new, generalises Theorem 6.4 of Daleckiĭ and Kreĭn [1], Chapter 3.

Proposition 3 is essentially due to Lazer [1]. It has been shown by Palmer [1] that, for *real* systems, there is still an exponential dichotomy if, more generally, (7) holds with $\delta = 0$ and $\inf|\det A(t)| > 0$.

Lecture 7. The connection between Lyapunov functions and exponential dichotomies was first considered by Maĭzel' [1]. The connection with ordinary dichotomies is also studied in Massera and Schäffer [3], Chapter 9. For Proposition 4 cf. Ostrowski and Schneider [1], Daleckiĭ and Kreĭn [1], Chapter 1, and Wimmer and Ziebur [1]. For the concept of controllability see, e.g., Kalman et al. [1].

Propositions 2.1, 3.3, and 7.2 state properties equivalent to an exponential dichotomy under the assumption of bounded growth. We pose the problem: what are the relations between these properties without this assumption?

Lecture 8. Propositions 1 and 3 are contained in Massera and Schäffer [3], Theorems 81.E and 103.A. For Proposition 4 see Coppel [2].

Lecture 9. Propositions 1 and 3 are due to Sacker and Sell [1], although their original proofs used algebraic topology. Proofs similar to those given here were found independently by Kato and Nakajima [1]. Further results are contained in Sacker and Sell [2], [3]. Proposition 4 is new. It would be desirable to have a formulation analogous to that of Proposition 1, in which the hypothesis of recurrence is omitted and the conclusion applies to each equation (1) in the ω-limit set of (2). Lemma 3 was suggested by Nakajima [1]. The application to Hahn's problem was pointed out in Coppel [4]. The result on orbital stability cannot be extended to almost periodic solutions which are not quasi-periodic; cf. Theorem 3.4 of Allaud and Thomas [1].

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